EVERY NATURAL NUMBER IS A SUM OF DISTINCT SEMIPRIME UNIT FRACTIONS

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Abstract. We prove that every natural number is a finite sum of distinct unit fractions whose denominators are *semiprimes* (products of two distinct primes). This is the $\omega = 2$ integer case of a problem of Erdős and Graham, stated only as a conjecture by Butler, Erdős and Graham (*Integers* **15** (2015), A51), who proved the $\omega = 3$ analogue. Counterintuitively the problem hardens as ω decreases — the induction's feed thins — so $\omega = 2$ is the hard case; our proof adapts the Butler–Erdős–Graham induction to this thin-feed regime, where the entire content of the induction step reduces to an explicit onset inequality $Y_0(N) \le \beta(N)$, proved for all $N \ge 10$ by Olson's addition theorem and elementary Chebyshev bounds. The same engine extends to the rationals: for every squarefree b, every a/b above an explicit threshold $\min\{B_{N_b}/6, \frac{1}{5}\}$ is $\omega = 2$ representable, unconditionally. As an application we give the first complete proof of the rational $\omega = 3$ statement of Erdős and Graham — every a/b with squarefree b is a sum of distinct *sphenic* unit fractions — which BEG15 left unpublished; a descent settles every $\omega \ge 3$. What remains open is the $\omega = 2$ regime below this threshold, which we reduce to an anti-concentration statement about an explicit subset-sum set.

1. Introduction

1.1. **Egyptian fractions with restricted denominators.** An *Egyptian fraction* representation of a positive rational a/b is an expression

$$\frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}, \qquad n_1 < n_2 < \cdots < n_k,$$

as a sum of *distinct* unit fractions. Such representations always exist (the greedy algorithm of Fibonacci produces one), and a rich vein of problems arises when one constrains the denominators n_i. Erdős and Graham [4] asked, for a squarefree denominator b, whether every a/b admits an Egyptian-fraction representation in which each n_i is a product of exactly ω distinct primes. This is recorded as problem #306 in Bloom's `erdosproblems.com` [1] and as D11 in Guy's *Unsolved Problems in Number Theory* [5].

For $\omega = 3$, Butler, Erdős and Graham [2] (henceforth *BEG15*) proved the integer case in full:

Theorem 1.1 (BEG15, Theorem 1)**.** *Any natural number can be written as an Egyptian fraction in which each denominator is the product of three distinct primes.*

In the same paper they remark that "similar arguments could be used" to handle $\omega \ge 4$, but for $\omega = 2$ they write only: "*We also conjecture that a similar result holds for* $\omega = 2$" ([2], p. 2). The difficulty is structural: as ω decreases, the combinatorial "feed" driving the BEG15 induction becomes thinner, and at $\omega = 2$ it degenerates to the very thinnest object the method can use — the set of subset sums of single-prime reciprocals. It is precisely this degeneration that BEG15 declined to push through; we isolate the exact mechanism, and our way past it, in §1.3.

1.2. **Main result.** We resolve the integer case of the $\omega = 2$ conjecture.

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Theorem 1.2. *Every natural number $m \geq 1$ can be written as a finite sum of distinct unit fractions*

$$m = \sum_i \frac{1}{n_i}, \qquad n_1 < n_2 < \cdots < n_k,$$

in which each denominator n_i is a product of two distinct primes (a semiprime).

Equivalently, the reciprocals $\{1/(pq) : p \neq q \text{ prime}\}$ form a universal additive basis for $\mathbb{N}$ in *distinct* unit fractions. This is the integer ($b = 1$) case of the Erdős–Graham problem above, and exactly the $\omega = 2$ statement conjectured in BEG15.

Theorem 1.2 is the engine; the rest of the paper drives it as far as it goes. §7 extends to all squarefree denominators above an explicit threshold (Theorems 7.1, 7.4); §8 proves the full rational $\omega = 3$ statement of Erdős and Graham (Theorem 8.1), the claim BEG15 left unpublished, by lifting the $\omega = 2$ threshold theorem, and a descent extends it to all $\omega \geq 3$ (Corollary 8.6); and §9 isolates the single remaining open statement as an anti-concentration property of an explicit subset-sum set. All results are unconditional. A reader wanting only the integer theorem may read §§2–6; §§7–8 are the rational extension, and §9 the open core.

A concrete instance for $m = 1$ was already known: Johnson [6] (reproduced in [5] and in BEG15) exhibited 1 as a sum of 48 unit fractions with semiprime denominators. What was missing — and what we supply — is the assertion for *all* natural numbers, by an effective but non-constructive argument paralleling the $\omega = 3$ case.

1.3. **Where BEG15 stops, and the new ingredient.** Since this paper pushes a known method through its hardest case, we state at the outset exactly where the published $\omega = 3$ argument halts and how we get past it.

The skeleton is BEG15's. We use the BEG15 machine unchanged: the splitting recursion

$$L^2(N+1) = L^1(N) + p_{N+1}\, L^2(N) \tag{S}$$

(Section 4), the propagation of a central contiguous block of subset sums by induction on N (Lemma 3.1), Olson's addition theorem [8] to control residues modulo the incoming prime $q = p_{N+1}$, and the Mertens-type growth $B_N \to \infty$ that makes the induction's covering intervals eventually swallow every integer. None of these ingredients is new.

The precise obstruction. What varies with ω is the *feed*: the leading term $L^{\omega-1}(N)$ of the recursion, which must realize every residue class modulo q. For $\omega \geq 3$ the feed $L^{\omega-1}(N)$ is already a rich subset-sum set, and BEG15 fill the target block by writing this term as shifted copies of an inner, residue-complete subset-sum set $P(S_{\dots})$ and merging the copies into one contiguous run by a bounded max-gap ("runs connect") estimate; this requires the inner set to be *both* residue-complete *and* low-gap. At $\omega = 2$ the feed collapses to the thinnest object the method admits, $L^1(N) = P(D_1(N))$, the subset sums of the N single-prime atoms P_N/p_i, and the inner set on which the BEG15 decomposition rests degenerates to a *single atom* — neither residue-complete nor low-gap. The runs-connect argument then has nothing to connect, and the $\omega = 3$ proof does not transcribe. This is precisely why BEG15 could write that "similar arguments" handle $\omega \geq 4$ (where the feed only gets richer) yet recorded $\omega = 2$ only as a *conjecture*.

The new ingredient. We do not try to restore internal structure to the thin feed. Instead we observe that in the $\omega = 2$ recursion the covering window is always at least as long as the entire feed range: its length is $r_{hi} - r_{lo} \geq \sigma^1(N) = \max L^1(N)$ (Lemma 4.2). An interval that long cannot lie strictly inside $(0, \sigma^1(N))$, so it must contain one of the feed endpoints 0 or $\sigma^1(N)$. Hence the feed $L^1(N)$ is used *directly* as the residue-complete object — its residue-completeness modulo q is just Olson applied to the single-prime atoms (Fact 4.4) — and the only thing that can fail is whether the feed attains every residue *soon enough*, i.e. within the available strip. All remaining difficulty is thereby concentrated into the single endpoint inequality

$$Y_0(N) \;\leq\; \beta(N),$$

with $Y_0(N)$ the residue-completeness onset radius of the feed and $\beta(N)$ the strip length (Section 5). Proving this for every $N \geq 10$ — exactly on a finite range, and by an elementary Chebyshev/monotonicity estimate beyond it — is the content the $\omega = 2$ case demands and that BEG15 did not supply.

1.4. **Scope and computations.**

- Theorem 1.2 settles the $b = 1$ integer case of the problem, and Section 7 extends the *same* engine to every squarefree b *above* an explicit threshold (Theorems 7.1, 7.4): a/b is unconditionally representable once $a/b \geq \min\{B_{N_b}/6, \frac{1}{5}\}$. What remains **open** is only the *below-threshold* bottom edge of the $\omega = 2$ problem — rationals with small numerator and large squarefree denominator, where the target falls below the central block into the low range of $L^2(N)$. This is the same end-of-interval difficulty BEG15 leave unresolved even for $\omega = 3$ ([2], p. 8: "the structure of $L_{n+3}(n)$ is not well understood at the ends of the interval"). We resolve it for $\omega = 3$ (Theorem 8.1, by a lift that bypasses the deep core) and isolate the remaining $\omega = 2$ deep core as one explicit anti-concentration statement (AC) (§9).
- The proof is partly *computational*, but only in a bounded and independently reproducible way. Three of its ingredients are established by exact finite computation: the base case at $N_0 = 10$; the inequality $Y_0(N) \leq \min\{\beta(N), \beta'(N)\}$ for $10 \leq N \leq 299$; and the residue-completeness of the feed (Fact 4.4) in the four cases $N = 10, 11, 12, 13$ where Olson's bound does not apply. Every such computation is exact-integer arithmetic; none uses floating point in a load-bearing way. The remaining ingredients are analytic but *elementary*: Olson's addition theorem, the elementary Chebyshev prime bounds, and the monotonicity of A_N. The rational extension of Section 7 adds only a finite 47-pair sliver check; its uniform threshold reuses the $N_0 = 10$ base case above (no separate large computation). We describe all computations and their independent re-verification in Section 6.

1.5. **Relation to other recent work.** Two recent preprints address *different* restrictions on the denominator and do not bear on the present problem: Czenky et al. [3] study representations using a *fixed finite* set of primes (motivated by fusion categories), and a recent preprint [7] studies denominators of the shape $p^a q^b$ for two *fixed* primes. Neither proves the $\omega = 2$ universal statement for varying prime denominators. As of this writing the $\omega = 2$ integer case appears unpublished, and the problem is listed as open [1].

2. Notation and the reduction ($\star$)

Throughout, p_n denotes the n-th prime ($p_1 = 2$, $p_2 = 3, \dots$), and

$$P_N = p_1 p_2 \cdots p_N$$

is the N-th primorial. We work with subset sums of two "atom" sets.

Definition 2.1. For $1 \leq r \leq N$ put

$$D_r(N) = \Big\{ \tfrac{P_N}{p_{i_1} \cdots p_{i_r}} : 1 \leq i_1 < \cdots < i_r \leq N \Big\},$$

the set of $\binom{N}{r}$ integers obtained from P_N by deleting r distinct prime factors. Given a finite set $X = \{x_1, \dots, x_m\}$ of integers, write

$$P(X) = \Big\{ \textstyle\sum_{i=1}^{m} \varepsilon_i x_i : \varepsilon_i \in \{0, 1\} \Big\}$$

for its set of subset sums. Put

$$L^r(N) = P\big(D_r(N)\big), \qquad \sigma^r(N) = \sum_{x \in D_r(N)} x,$$

so that $\sigma^r(N) = \max L^r(N)$ and (since $0 \in L^r(N)$ and $L^r(N)$ is symmetric about $\sigma^r(N)/2$) $L^r(N) \subseteq [0, \sigma^r(N)]$.

We use the two cases $r = 1$ and $r = 2$:

$$D_1(N) = \{P_N/p_i : 1 \le i \le N\}, \qquad D_2(N) = \{P_N/(p_ip_j) : 1 \le i < j \le N\}.$$

We call $D_1(N)$ the *feed* and $L^1(N)$ the *feed sums*; $D_2(N)$ is the *semiprime atom set* for the first N primes.

Remark 2.2. Dividing an element $P_N/(p_ip_j) \in D_2(N)$ by P_N gives the unit fraction $1/(p_ip_j)$ with semiprime denominator. Thus subset sums of $D_2(N)$ correspond exactly to sums of distinct semiprime unit fractions whose primes lie among the first N. This yields the basic reduction.

Proposition 2.3 (reduction $(\star)$)**.** *A positive rational m is a sum of distinct unit fractions with semiprime denominators using primes among the first N if and only if*

$$(\star) \qquad P_N\, m \in L^2(N).$$

In particular, if a natural number m satisfies $P_N m \in L^2(N)$ for some N, then m has a representation as in Theorem 1.2.

Proof. A subset $S \subseteq D_2(N)$ with $\sum S = P_N m$ corresponds, after dividing by P_N, to a set of distinct semiprime reciprocals $1/(p_ip_j)$ summing to m; the denominators are distinct semiprimes since the atoms $P_N/(p_ip_j)$ are distinct. Conversely a representation $m = \sum 1/(p_ip_j)$ with primes among the first N scales to a subset of $D_2(N)$ summing to $P_N m$. □

It is convenient to normalize $\sigma^2(N)$ by P_N. Set

$$A_N := \sum_{i \le N} \frac{1}{p_i}, \qquad B_N := \frac{\sigma^2(N)}{P_N} = \sum_{1 \le i < j \le N} \frac{1}{p_ip_j} = \frac{1}{2}\left(A_N^2 - \sum_{i \le N} \frac{1}{p_i^2}\right).$$

By Mertens' theorem $A_N \to \infty$, while $\sum_i p_i^{-2}$ converges; hence $B_N \to \infty$, and indeed $B_N \sim \frac{1}{2}(\log\log N)^2$. Below we use only that A_N is increasing and unbounded.

Table 1 collects the recurring notation.

symbol	meaning	defined
p_i	the i-th prime ($p_1 = 2$); $q = p_{N+1}$	§2
P_N	primorial $p_1 \cdots p_N$	§2
$D_r(N)$	atoms $P_N/(p_{i_1} \cdots p_{i_r})$ (r primes deleted)	Def. 2.1
$L^r(N)$	subset sums of $D_r(N)$; $\sigma^r(N) = \max L^r(N)$	Def. 2.1
A_N	$\sum_{i \le N} 1/p_i$	$(\star)$ ff.
B_N	$\sigma^2(N)/P_N = \sum_{i<j} 1/(p_ip_j)$ (normalized)	§2
$\beta(N), \beta'(N)$	integer strip lengths $a_{N+1} - q\,a_N$, etc.	Def. 5.1
$Y_0(N)$	residue-completeness onset radius of the feed	Def. 5.1
γ_N, γ_∞	gap-free floor of $L^2(N)$; its uniform bound	§7
$\gamma_{\mathrm{ex}}(M, r)$	gap-free floor with prime r deleted	§8
w_k	floor-recurrence cost $= Y_0(k-1)/P_{k-1}$	(2)
$B_k(N)$	$\sum_{i_1 < \cdots < i_k} 1/(p_{i_1} \cdots p_{i_k})$; $B_2 = B_N$	§8
$B_\Pi(M, r)$	B_2 over the primes $\{p_1, \ldots, p_M\} \setminus \{r\}$	§8
N_b	index of the largest prime factor of b	§7

TABLE 1. Recurring notation.

3. The induction skeleton

The engine of the proof is the following block-containment statement, the $\omega = 2$ analogue of BEG15's Lemma 1.

Lemma 3.1 (central block)**.** *For every $N \geq 10$,*

$$L^2(N) \;\supseteq\; \left[\left\lceil \tfrac{1}{6}\sigma^2(N)\right\rceil,\ \left\lfloor \tfrac{5}{6}\sigma^2(N)\right\rfloor\right] \cap \mathbb{Z}.$$

That is, $L^2(N)$ contains every *integer in the central block $\left[\frac{1}{6}\sigma^2(N), \frac{5}{6}\sigma^2(N)\right]$.*

Lemma 3.1 is proved by induction on N, with:

- **Base case** $N_0 = 10$ (Proposition 3.2, a direct exact computation), and
- **Induction step** $N \to N+1$, valid for all $N \geq 10$ (Section 4).

We first record the base case and then deduce Theorem 1.2, deferring the step to the next section.

Proposition 3.2 (base case)**.** *With $N_0 = 10$,*

$$P_{10} = 6{,}469{,}693{,}230, \qquad \sigma^2(10) = 6{,}166{,}988{,}769,$$

and

$$L^2(10) \;\supseteq\; \left[1{,}027{,}831{,}462,\ 5{,}139{,}157{,}307\right] \cap \mathbb{Z} = \left[\left\lceil \tfrac{1}{6}\sigma^2(10)\right\rceil, \left\lfloor \tfrac{5}{6}\sigma^2(10)\right\rfloor\right] \cap \mathbb{Z},$$

with no integer of the block missing.

Proof. This is verified by exact subset-sum reachability over the $\binom{10}{2} = 45$ atoms of $D_2(10)$; see Section 6 for the method (a packed bitset of $\sigma^2(10)+1$ bits, $\approx 771\,\mathrm{MB}$) and its independent re-verification. Every integer in the stated block is a subset sum of $D_2(10)$; 0 integers are missing. □

Remark 3.3 (why the base is $N_0 = 10$, not 9)**.** One might hope to seed at $N_0 = 9$, where $L^2(9)$ also covers its central block [33,520,589, 167,602,941] exactly. However the induction step $N \to N+1$ requires the inequality $Y_0(N) \leq \beta(N)$ of Section 5, and this *fails* at $N = 9$: $Y_0(9) = 91{,}525{,}280 > \beta(9) = 55{,}734{,}381$. The inequality $Y_0(N) \leq \beta(N)$ holds for all $N \geq 10$, so seeding directly at $N_0 = 10$ — where the step is valid from the start — repairs this and the induction runs cleanly upward.

We isolate the elementary overlap estimate that drives the interval-covering argument.

Proposition 3.4 (explicit overlap)**.** *For every $N \geq 1$ the sequence B_N is strictly increasing, and for every $N \geq 10$,*

$$B_{N+1} \leq 5B_N, \qquad \textit{equivalently} \qquad B_{N+1} - B_N \leq 4B_N.$$

Proof. The increment is

$$B_{N+1} - B_N = \sum_{i \leq N} \frac{1}{p_i\, p_{N+1}} = \frac{A_N}{p_{N+1}} > 0,$$

since the only pairs counted by B_{N+1} and not by B_N are $\{i, N+1\}$ with $i \leq N$. Hence B_N is strictly increasing, and by direct computation $B_{10} = 0.9532\ldots > \frac{1}{4}$, so $B_N \geq B_{10} > \frac{1}{4}$ for all $N \geq 10$; thus $4B_N > 1$. On the other hand $p_i \geq 2$ gives $A_N = \sum_{i \leq N} 1/p_i \leq N/2$, while the $(N+1)$-st prime satisfies $p_{N+1} \geq N+2$ (among $1, \ldots, p_{N+1}$ there are exactly $N+1$ primes and 1 is not prime, so $p_{N+1} \geq N+2$). Therefore

$$B_{N+1} - B_N = \frac{A_N}{p_{N+1}} \leq \frac{N/2}{N+2} < \frac{1}{2} < 1 < 4B_N,$$

which is the claim. □

Proof of Theorem 1.2, assuming Lemma 3.1. Fix a natural number $m \geq 1$. By ($\star$) (Proposition 2.3), m is representable as soon as $P_N m \in L^2(N)$ for some N. By Lemma 3.1, $L^2(N)$ contains every integer in $[\frac{1}{6}\sigma^2(N), \frac{5}{6}\sigma^2(N)]$; since $P_N m$ is an integer, it suffices that

$$\tfrac{1}{6}\,\sigma^2(N) \;\leq\; P_N m \;\leq\; \tfrac{5}{6}\,\sigma^2(N), \quad \text{equivalently} \quad \tfrac{1}{6}B_N \;\leq\; m \;\leq\; \tfrac{5}{6}B_N,$$

for some $N \geq 10$ (dividing by P_N and using $B_N = \sigma^2(N)/P_N$).

Consider the intervals $I_N := [\frac{1}{6}B_N, \frac{5}{6}B_N]$ for $N \geq 10$. The right endpoint of I_N is at or above the left endpoint of I_{N+1} exactly when $\frac{5}{6}B_N \geq \frac{1}{6}B_{N+1}$, i.e. $B_{N+1} \leq 5B_N$, which holds for all $N \geq 10$ by Proposition 3.4. Consecutive intervals therefore overlap, and since B_N is increasing and unbounded ($B_N \to \infty$), the union $\bigcup_{N\geq 10} I_N$ is the half-line $[\frac{1}{6}B_{10}, \infty)$. As $B_{10} = \sigma^2(10)/P_{10} = 6{,}166{,}988{,}769/6{,}469{,}693{,}230 = 0.9532\ldots$ gives $\frac{1}{6}B_{10} = 0.1589 < 1$, this half-line contains every integer $m \geq 1$. (For instance $m = 1$ first lies in I_N at $N = 17$, where $B_{17} = 1.2162\ldots$ gives $I_{17} = [0.2027, 1.0135] \ni 1$.) Every integer $m \geq 1$ thus lies in some I_N, hence is representable. □

Remark 3.5. As in BEG15, the bound on N needed to represent a given m is enormous and the argument is non-constructive: m enters I_N only once $B_N \sim \frac{1}{2}(\log\log N)^2 \gtrsim \frac{6}{5}m$, i.e. at N of order $\exp\exp(O(\sqrt{m}))$. Theorem 1.2 asserts existence of a representation, not a small one.

4. The induction step $N \to N+1$

We now prove the step in Lemma 3.1: assuming

$$L^2(N) \supseteq \big[a_N, b_N\big] \cap \mathbb{Z}, \qquad a_N = \big\lceil \tfrac{1}{6}\sigma^2(N) \big\rceil, \quad b_N = \big\lfloor \tfrac{5}{6}\sigma^2(N) \big\rfloor,$$

we deduce the same with $N+1$ in place of N, for every $N \geq 10$.

4.1. **The splitting recursion.** Partitioning $D_2(N+1)$ according to whether the largest available prime p_{N+1} is deleted gives the disjoint union

$$D_2(N+1) = D_1(N) \;\sqcup\; p_{N+1} \cdot D_2(N). \tag{R}$$

Indeed, an element of $D_2(N+1)$ is P_{N+1} divided by two distinct primes among $p_1, \ldots, p_{N+1}$. If neither deleted prime is p_{N+1} we get $P_{N+1}/(p_i p_j) = p_{N+1} \cdot P_N/(p_i p_j) \in p_{N+1} D_2(N)$; if one deleted prime is p_{N+1} we get $P_{N+1}/(p_{N+1} p_i) = P_N/p_i \in D_1(N)$. Taking subset sums of a disjoint union, and recalling $P(A \sqcup B) = P(A) + P(B)$ (sumset),

$$L^2(N+1) = L^1(N) + p_{N+1}\, L^2(N). \tag{S}$$

This is the $r = 2$ instance of BEG15's recursion (1). Likewise summing all of $D_2(N+1)$ gives $\sigma^2(N+1) = \sigma^1(N) + p_{N+1}\sigma^2(N)$.

4.2. **Reformulating the step as a covering problem.** The objects in play, all at level N: the incoming prime $q = p_{N+1}$; the inductive block $[a_N, b_N] \subseteq L^2(N)$ and its q-scaling $[r_{lo}, r_{hi}] = q[a_N, b_N]$; the feed range $[0, \sigma^1(N)]$; and, for each target x, the feed window $W(x)$ defined below. By the induction hypothesis $L^2(N) \supseteq [a_N, b_N]$, so $p_{N+1}L^2(N) = q\,L^2(N)$ contains *every multiple of* q in $[r_{lo}, r_{hi}]$, where

$$r_{lo} = q\,a_N = q\big\lceil \tfrac{1}{6}\sigma^2(N) \big\rceil, \qquad r_{hi} = q\,b_N = q\big\lfloor \tfrac{5}{6}\sigma^2(N) \big\rfloor.$$

By (S), an integer x is in $L^2(N+1)$ provided there is a feed sum $u \in L^1(N)$ with $u \equiv x \pmod q$ and $r_{lo} \leq x - u \leq r_{hi}$ (then $x = u + (x-u)$ with $x-u$ a multiple of q in $[r_{lo}, r_{hi}]$, hence $x - u \in qL^2(N)$). Equivalently, $x \in L^2(N+1)$ if the feed window

$$W(x) := [\,x - r_{hi},\, x - r_{lo}\,] \cap [0, \sigma^1(N)]$$

contains an element of $L^1(N)$ congruent to x mod q. We must show this holds for every integer x in the target block $[a_{N+1}, b_{N+1}]$.

4.3. **The window always spans an endpoint of the feed.** *This is the observation the $\omega = 2$ case turns on.* The window $[x-r_{hi}, x-r_{lo}]$ has length $r_{hi}-r_{lo}$. We show this length is at least $\sigma^1(N)$, so that the window cannot lie strictly inside the open feed range $(0, \sigma^1(N))$: an interval of length $\geq \sigma^1$ cannot be a subset of an open interval of length σ^1. Hence $W(x) = [x-r_{hi}, x-r_{lo}] \cap [0, \sigma^1(N)]$ always contains one of the two endpoints 0 or $\sigma^1(N)$. The inequality $r_{hi}-r_{lo} \geq \sigma^1(N)$ is the second seam inequality, proved with large margin in Lemma 4.2. This splits the target block into three contiguous pieces (Figure 1).

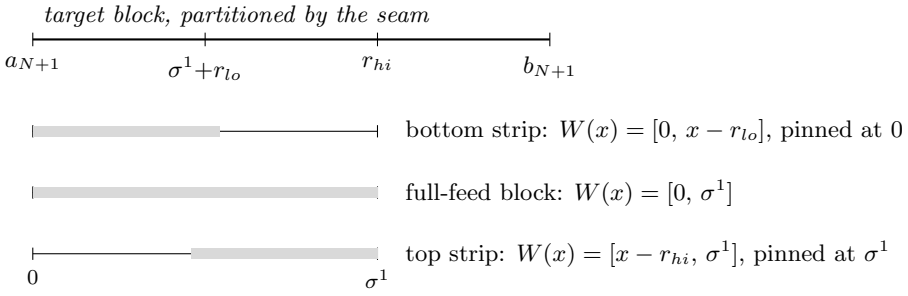

FIGURE 1. The induction step covers the target block in three pieces. The covering window $W(x)$ has length $\geq \sigma^1$, so it cannot fit strictly inside the feed range $(0, \sigma^1)$ and must meet an endpoint: pinned at 0 in the bottom strip, at σ^1 in the top strip, and equal to the whole feed in between. Each piece is covered once the feed $L^1(N)$ hits every residue mod q soon enough (Proposition 4.3).

Full feed. For $x \in [\sigma^1(N) + r_{lo},\, r_{hi}]$ the window is the whole feed range, $W(x) = [0, \sigma^1(N)]$. Here x is covered as soon as the feed sums $L^1(N)$ are *residue-complete* modulo q, i.e. $L^1(N) \bmod q = \mathbb{Z}_q$; this is Fact 4.4 below.

Bottom strip. For $x \in [a_{N+1},\, \sigma^1(N) + r_{lo})$ the window touches 0: $W(x) = [0,\, x - r_{lo}]$, and $x - r_{lo} \geq a_{N+1} - r_{lo} =: \beta(N)$. Thus $W(x) \supseteq [0, \beta(N)]$. So x is covered for *all* such x provided the feed sums lying in the initial segment $[0, \beta(N)]$ already hit every residue mod q, i.e. provided $\beta(N) \geq Y_0(N)$, where $Y_0(N)$ (Definition 5.1) is the smallest y for which $L^1(N) \cap [0, y]$ is residue-complete.

Top strip. For $x \in (r_{hi},\, b_{N+1}]$ the window touches $\sigma^1(N)$: $W(x) = [x - r_{hi},\, \sigma^1(N)]$. The feed $L^1(N)$ is symmetric under $u \mapsto \sigma^1(N) - u$ (subset complementation in $D_1(N)$), so the top "onset radius" measured from $\sigma^1(N)$ downward equals $Y_0(N)$. The available top length is shortest at the right end $x = b_{N+1}$, where it equals

$$\beta'(N) := \sigma^1(N) - \big(b_{N+1} - r_{hi}\big) = \sigma^1(N) - b_{N+1} + q\, b_N,$$

so the top strip is covered provided $Y_0(N) \leq \beta'(N)$. We do *not* claim $\beta'(N) = \beta(N)$: the two strip lengths can differ by a bounded rounding term. What we use is that *both* dominate the same explicit integer lower bound,

$$\beta(N) \;\geq\; \tfrac{1}{6}\sigma^1(N) - q, \qquad \beta'(N) \;\geq\; \tfrac{1}{6}\sigma^1(N) - q \tag{LB}$$

(Lemmas 5.2 and 5.3). Hence the top strip is covered under the *same* sufficient condition $Y_0(N) \leq \frac{1}{6}\sigma^1(N) - q$ as the bottom strip, and it is this common bound that Theorem 5.4 establishes.

Remark 4.1 (the step at $N = 10$, $q = 31$, normalized by P_{10})**.** Concretely (Figure 1): $\sigma^1(10) = 1.5334\, P_{10}$, and the target block is $[5.18,\, 25.90]\, P_{10}$, split at $\sigma^1 + r_{lo} = 6.46\, P_{10}$ and $r_{hi} = 24.63\, P_{10}$ into the bottom strip, full-feed block, and top strip. A target $x = 5.5\, P_{10}$ lies in the bottom strip; its window is $W(x) = [0,\, 0.575\, P_{10}]$, pinned at 0, and is covered because the feed reaches every residue mod 31 already by $Y_0(10) = 0.232\, P_{10}$, within the available $\beta(10) = 0.256\, P_{10}$. The margin 0.024 is the tightest in the entire induction (Appendix A).

4.4. **The seams cover, for every $N \geq 10$.** For these three pieces to tile the target block $[a_{N+1}, b_{N+1}]$ we need the ordering

$$r_{lo} \;<\; a_{N+1} \;\leq\; \sigma^1(N) + r_{lo} \;\leq\; r_{hi} \;\leq\; b_{N+1} \;<\; r_{hi} + \sigma^1(N). \tag{Seam}$$

We prove all of (Seam) *analytically*, for every $N \geq 10$, with explicit margins.

Lemma 4.2 (seam ordering)**.** *For every $N \geq 10$ the chain* (Seam) *holds; consequently the three pieces cover $[a_{N+1}, b_{N+1}] \cap \mathbb{Z}$ exactly.*

Proof. Write $q = p_{N+1}$, $\sigma^1 = \sigma^1(N)$, $\sigma^2 = \sigma^2(N)$, and recall $\sigma^2(N+1) = \sigma^1 + q\,\sigma^2$, $a_M = \lceil \sigma^2(M)/6 \rceil$, $b_M = \lfloor 5\sigma^2(M)/6 \rfloor$, $r_{lo} = q\,a_N$, $r_{hi} = q\,b_N$. We use two elementary facts valid for $N \geq 10$:

(P1) $\sigma^1 \geq \frac{6}{5}$ (indeed $\sigma^1 \geq \sigma^1(10) \approx 9.9 \times 10^9$);

(P2) $\sigma^1 > 6q$. For $\sigma^1/q = P_N A_N / p_{N+1}$, Bertrand's postulate $p_{N+1} < 2p_N$ and $P_N = p_N P_{N-1}$ give $P_N/p_{N+1} > P_{N-1}/2$, so with $A_N \geq A_{10} > 1.53$ and $P_{N-1} \geq P_9 = 223{,}092{,}870$ we get $\sigma^1/q > 1.53 \cdot P_9/2 > 10^8 \gg 6$.

$r_{lo} < a_{N+1}$. Using $\lceil t \rceil \geq t$ and $\lceil t \rceil \leq t+1$,

$$a_{N+1} = \left\lceil \tfrac{\sigma^1 + q\sigma^2}{6} \right\rceil \geq \tfrac{\sigma^1}{6} + \tfrac{q\sigma^2}{6}, \qquad r_{lo} = q\lceil \tfrac{\sigma^2}{6} \rceil \leq \tfrac{q\sigma^2}{6} + q,$$

so $a_{N+1} - r_{lo} \geq \frac{1}{6}\sigma^1 - q > 0$ by (P2).

$a_{N+1} \leq \sigma^1 + r_{lo}$. Here $a_{N+1} \leq \frac{\sigma^1}{6} + \frac{q\sigma^2}{6} + 1$ and $\sigma^1 + r_{lo} \geq \sigma^1 + \frac{q\sigma^2}{6}$, so the difference is $\geq \frac{5}{6}\sigma^1 - 1 > 0$ by (P1).

$\sigma^1 + r_{lo} \leq r_{hi}$ *(the key seam).* Now $r_{hi} - r_{lo} = q(b_N - a_N) \geq q\big(\frac{5}{6}\sigma^2 - 1 - \frac{1}{6}\sigma^2 - 1\big) = q\big(\frac{2}{3}\sigma^2 - 2\big)$, so it suffices that $\sigma^1 \leq q(\frac{2}{3}\sigma^2 - 2)$; dividing by P_N this is

$$A_N \;\leq\; \tfrac{2}{3} q\, B_N - \tfrac{2q}{P_N}.$$

Since $2B_N = A_N^2 - S_N$ with $S_N := \sum_{i \leq N} p_i^{-2} < \sum_p p^{-2} < 0.4523$ and $A_N \geq A_{10} > 1.53$,

$$\frac{A_N}{B_N} = \frac{2}{A_N - S_N/A_N} \leq \frac{2}{1.53 - 0.4523/1.53} < 1.63.$$

As $q \geq p_{11} = 31$, we get $\frac{2}{3} q B_N - A_N \geq (\frac{2}{3} \cdot 31 - 1.63) B_N > 18 B_N > 18 B_{10} > 17$, which dwarfs $2q/P_N < 2 \cdot 31/P_{10} < 10^{-7}$. Hence $r_{hi} - r_{lo} - \sigma^1 > 17 P_N > 0$, which in particular gives $r_{hi} - r_{lo} \geq \sigma^1$ (used in Section 4).

$r_{hi} \leq b_{N+1}$. Symmetrically $r_{hi} = q\lfloor 5\sigma^2/6 \rfloor \leq \frac{5}{6} q\sigma^2$ and $b_{N+1} \geq \frac{5}{6}\sigma^1 + \frac{5}{6} q\sigma^2 - 1$, so $b_{N+1} - r_{hi} \geq \frac{5}{6}\sigma^1 - 1 > 0$ by (P1).

$b_{N+1} < r_{hi} + \sigma^1$. By definition $\beta'(N) = \sigma^1 - (b_{N+1} - r_{hi})$, so this is $\beta'(N) > 0$, which follows from Lemma 5.3 and (P2): $\beta'(N) \geq \frac{1}{6}\sigma^1 - q > 0$.

The displayed bounds partition $[a_{N+1}, b_{N+1}]$ into the bottom strip $[a_{N+1}, \sigma^1 + r_{lo})$, the full-feed block $[\sigma^1 + r_{lo}, r_{hi}]$, and the top strip $(r_{hi}, b_{N+1}]$. □

Proposition 4.3 (reduction of the step)**.** *Fix $N \geq 10$ and assume $L^2(N) \supseteq [a_N, b_N] \cap \mathbb{Z}$. If*

(i) *the feed sums $L^1(N)$ are residue-complete modulo $q = p_{N+1}$, and*
(ii) *$Y_0(N) \leq \min\{\beta(N), \beta'(N)\}$; by* (LB) *it suffices that $Y_0(N) \leq \frac{1}{6}\sigma^1(N) - q$,*

then $L^2(N+1) \supseteq [a_{N+1}, b_{N+1}] \cap \mathbb{Z}$.

It remains to establish (i) and (ii) for all $N \geq 10$. Ingredient (i) is Fact 4.4; ingredient (ii) is the analytic heart, proved in Section 5.

4.5. **Ingredient (i): residue-completeness of the feed.**

Fact 4.4 (feed residue-completeness)**.** For every $N \geq 10$ the feed sums $L^1(N) = P(D_1(N))$ are residue-complete modulo $q = p_{N+1}$; that is, $L^1(N) \bmod q = \mathbb{Z}_q$.

Proof. The atoms $D_1(N) = \{P_N/p_i : 1 \leq i \leq N\}$ reduce modulo $q = p_{N+1}$ to N *distinct, nonzero* residues: nonzero because q exceeds $p_1, \ldots, p_N$ and does not divide P_N/p_i; distinct because P_N is a unit mod q and the inverses $p_i^{-1} \bmod q$ are distinct, so $P_N/p_i \equiv P_N\, p_i^{-1}$ are distinct. By Olson's theorem (Theorem 4.5) these N residues have $P(D_1(N) \bmod q) = \mathbb{Z}_q$ whenever $q = p_{N+1} < (N^2+3)/4$. The actual primes satisfy this for all $N \geq 14$: directly for $14 \leq N \leq 27$, and for $N \geq 28$ by the elementary bound $p_{N+1} < 2(N+1)\log(N+1) < (N^2+3)/4$ (the second inequality first holds at $N = 28$, where $2 \cdot 29 \log 29 = 195.3 < 196.75$, and then for all larger N as the gap is increasing). Olson's hypothesis *fails* only for $N \leq 13$ (at $N = 13$, $p_{14} = 43 = (13^2+3)/4$, not strictly less); among these the proof uses only $N \geq 10$, and the four cases $N = 10, 11, 12, 13$ ($q \in \{31, 37, 41, 43\}$) are settled by exact subset-sum computation (Section 6). □

Theorem 4.5 (Olson [8]; BEG15 Theorem 2)**.** *Let X be a set of distinct nonzero residues modulo a prime q. If $q < (|X|^2+3)/4$, then $P(X) = \mathbb{Z}_q$.*

Equivalently (the form we use below), a set of more than $\sqrt{4q-3}$ distinct nonzero residues modulo q has subset sums equal to all of $\mathbb{Z}_q$.

5. The analytic heart: $Y_0(N) \leq \beta(N)$

Definition 5.1. The *residue-completeness onset radius* of the feed is

$$Y_0(N) := \max_{r \in \mathbb{Z}_q} \min\bigl\{\textstyle\sum S : \ S \subseteq D_1(N),\ \sum S \equiv r \pmod q\bigr\}, \qquad q = p_{N+1},$$

i.e. the smallest y such that $L^1(N) \cap [0, y]$ meets every residue class modulo q. (By Fact 4.4 the inner minima are finite, so $Y_0(N)$ is well defined.) Recall from §4 the two strip lengths

$$\beta(N) := a_{N+1} - q\, a_N = \bigl\lceil \tfrac{1}{6}\sigma^2(N+1)\bigr\rceil - q\bigl\lceil \tfrac{1}{6}\sigma^2(N)\bigr\rceil, \qquad \beta'(N) := \sigma^1(N) - b_{N+1} + q\, b_N.$$

Lemma 5.2. $\beta(N) \ \geq \ \dfrac{\sigma^1(N)}{6} - q.$

Proof. Recall $\beta(N) = \lceil \frac{1}{6}\sigma^2(N+1)\rceil - q\lceil \frac{1}{6}\sigma^2(N)\rceil$ and $\sigma^2(N+1) = \sigma^1(N) + q\,\sigma^2(N)$. Bound the two terms separately. For the first, $\lceil t \rceil \geq t$ gives

$$\left\lceil \tfrac{1}{6}\sigma^2(N+1)\right\rceil \ \geq \ \tfrac{1}{6}\sigma^2(N+1) \ = \ \tfrac{1}{6}\sigma^1(N) + \tfrac{q}{6}\sigma^2(N).$$

For the second, $\lceil t \rceil \leq t + 1$ gives

$$q\left\lceil \tfrac{1}{6}\sigma^2(N)\right\rceil \ \leq \ q\Bigl(\tfrac{1}{6}\sigma^2(N) + 1\Bigr) \ = \ \tfrac{q}{6}\sigma^2(N) + q.$$

Subtracting, the $\frac{q}{6}\sigma^2(N)$ terms cancel and $\beta(N) \geq \frac{1}{6}\sigma^1(N) - q$. □

Lemma 5.3. $\beta'(N) \ \geq \ \dfrac{\sigma^1(N)}{6} - q.$

Proof. With $\sigma^2(N+1) = \sigma^1(N) + q\,\sigma^2(N)$ we have $b_{N+1} \leq \frac{5}{6}\sigma^1(N) + \frac{5q}{6}\sigma^2(N)$, while $q\, b_N \geq \frac{5q}{6}\sigma^2(N) - q$. Hence $\beta'(N) = \sigma^1(N) - b_{N+1} + q\, b_N \geq \frac{1}{6}\sigma^1(N) - q$. □

Theorem 5.4. *For all $N \geq 10$, $Y_0(N) \leq \min\{\beta(N), \beta'(N)\}$. Indeed for all $N \geq 300$ the stronger bound $Y_0(N) \leq \frac{1}{6}\sigma^1(N) - q$ holds, which by Lemmas 5.2–5.3 implies the former; the finite range $10 \leq N \leq 299$ is checked exactly.*

We prove this in two regimes: an exact finite range and an elementary analytic tail.

5.1. **Exact range $10 \le N \le 299$.** For each such N the quantity $Y_0(N)$ is computed *exactly* by a Dijkstra-style shortest-path computation over the residue group $\mathbb{Z}_q$: maintain, for each residue r, the minimal subset sum of $D_1(N)$ achieving r, updating atom by atom. The state space has size $q = p_{N+1} \le p_{300} = 1{,}987$ in this range, so the computation is polynomial (no exponential enumeration) and uses exact integers. One then checks $Y_0(N) \le \min\{\beta(N), \beta'(N)\}$ directly (both strip lengths are computed exactly from the primorials). All 290 values pass; see Section 6. (We note $Y_0(9) > \beta(9)$, consistent with Remark 3.3.)

5.2. **Elementary analytic tail $N \ge 300$.** Let $q = p_{N+1}$ and put $c_0 := \lfloor \sqrt{4q-3} \rfloor + 1$, so that $q < (c_0^2+3)/4$ strictly. Let C denote the c_0 *smallest* atoms of the feed, $C = \{P_N/p_i : i = N - c_0 + 1, \dots, N\}$. Their residues mod q are distinct and nonzero (as in Fact 4.4), and $|C| = c_0$ with $q < (c_0^2+3)/4$; for $N \ge 300$ one has $c_0 \le N$, so $C \subseteq D_1(N)$. By Olson's theorem $P(C) = \mathbb{Z}_q$, so every residue is achieved by a subset of C, whose total is at most $\sigma(C) := \sum_{i=N-c_0+1}^{N} P_N/p_i$. Therefore

$$Y_0(N) \;\le\; \sigma(C). \tag{O}$$

Dividing (O) by P_N, the desired bound $\sigma(C) \le \frac{1}{6}\sigma^1(N) - q$ is equivalent to

$$\Sigma(N) \;:=\; \sum_{i=N-c_0+1}^{N} \frac{1}{p_i} \;+\; \frac{q}{P_N} \;\le\; \frac{A_N}{6}. \tag{Tail}$$

We close (Tail) for all $N \ge 300$ using only the elementary Chebyshev prime bounds

$$L(k) := \tfrac{1}{2}k\log k \;<\; p_k \;<\; 2k\log k =: U(k) \qquad (k \ge 6), \tag{Ch}$$

which are provable by the binomial-coefficient method and which we verify directly for the (few) small indices used below, together with the monotonicity of A_N (explicit forms of all these prime estimates are classical, [9]).

Lemma 5.5 (elementary certified tail)**.** *For all $N \ge 300$, inequality* (Tail) *holds. Consequently $\sigma(C) \le \frac{1}{6}\sigma^1(N) - q$ and $Y_0(N) \le \frac{1}{6}\sigma^1(N) - q$.*

Proof. Write $g(N) := 2\sqrt{U(N+1)} + 1$.

Lower bound for the right side. A_N is strictly increasing (Proposition 3.4), so $A_N \ge A_{300}$; and $A_{300} = \sum_{i \le 300} 1/p_i = 2.2909\ldots$ is a finite sum, giving $A_N/6 \ge A_{300}/6 = 0.3818\ldots$

Upper bound for the left side. The c_0 summands are each $\le 1/p_{N-c_0}$, so the feed tail is $\le c_0/p_{N-c_0}$. By (Ch), $q = p_{N+1} < U(N+1)$, hence $c_0 \le 2\sqrt{q} + 1 < 2\sqrt{U(N+1)} + 1 = g(N)$; and since L is increasing and $c_0 < g(N)$, $p_{N-c_0} > L(N-c_0) \ge L(N-g(N))$. Thus the feed tail is $< \tau(N) := g(N)/L(N-g(N))$. The carry is $q/P_N \le U(N+1)/2^N =: \varepsilon(N)$, with $\varepsilon(300) < 3436/2^{300} < 10^{-86}$. Hence $\Sigma(N) < \overline{\Sigma}(N) := \tau(N) + \varepsilon(N)$.

Monotonicity. ε is decreasing. For τ, with $D := N - g(N)$,

$$(\log\tau)' = \frac{g'(N)}{g(N)} - \big(1 - g'(N)\big)\frac{L'(D)}{L(D)} \;\le\; \frac{g'(N)}{g(N)} - \frac{1 - g'(N)}{D},$$

using $L'(D)/L(D) = (\log D + 1)/(D\log D) \ge 1/D$ and $1 - g'(N) > 0$. Since $D + g(N) = N$, the right side is < 0 exactly when $g'(N)\,(D + g(N)) < g(N)$, i.e. $g'(N) < g(N)/N$. From $g(N) = 2\sqrt{U(N+1)} + 1$ with $U(k) = 2k\log k$ one computes $g'(N) = 2(\log(N+1)+1)/\sqrt{U(N+1)}$, which is decreasing; at $N = 300$, $g'(300) = 13.41/58.62 < 0.23$ while $g(300)/300 > 118/300 > 0.39$, and the ratio $g'(N)/(g(N)/N) \to \frac{1}{2}$. Hence $g'(N) < g(N)/N$ throughout $[300, \infty)$, so $(\log\tau)' < 0$ and $\overline{\Sigma}$ is decreasing there.

Base evaluation at $N = 300$. $U(301) = 602\log 301 < 3436$, so $g(300) < 2\sqrt{3436} + 1 < 118.3$, $D(300) > 181.7$, and $L(D(300)) > \frac{1}{2} \cdot 181.7 \cdot \log 181.7 > 472$; whence $\tau(300) < 118.3/472 < 0.2507$, and adding $\varepsilon(300) < 10^{-86}$ gives $\overline{\Sigma}(300) < 0.2507$.

Conclusion. For all $N \geq 300$,

$$\Sigma(N) < \overline{\Sigma}(N) \leq \overline{\Sigma}(300) < 0.2507 < 0.3818 = A_{300}/6 \leq A_N/6,$$

which is (Tail), with margin > 0.13. Dividing through by P_N in (O) gives $\sigma(C) \leq \frac{1}{6}\sigma^1(N) - q$ and hence $Y_0(N) \leq \frac{1}{6}\sigma^1(N) - q$. □

Lemma 5.5 for $N \geq 300$ together with the exact range $10 \leq N \leq 299$ proves Theorem 5.4.

5.3. **Why a worst-case bound does not suffice.** The exact finite range cannot be replaced by a crude worst-case bound. Olson's threshold $(m^2+3)/4$ is sharp (attained by $\{\pm 1, \pm 2, \dots\}$), so for $10 \leq N \leq 13$ the feed lies below the worst-case guarantee yet is residue-complete because its particular residues $\{P_N p_i^{-1} \bmod q\}$ are far from worst-case — a finite, set-specific fact, certified by an explicit representative per residue. No equidistribution estimate enters the analytic part either: for $N \geq 300$ the *sum* of the $c_0 \approx 2\sqrt{q}$ smallest atoms is already $\leq \beta(N)$ by the Chebyshev/monotonicity bound of §5.2.

6. Computational verification

Several ingredients are established by exact finite computation. All use exact-integer arithmetic; no floating-point value is load-bearing. (Floating point appears only in the monotone-envelope tail of §5.2, where the relevant inequalities carry margins far exceeding any rounding error.)

Base case at $N_0 = 10$ (Proposition 3.2). The set $D_2(10)$ has $\binom{10}{2} = 45$ atoms summing to $\sigma^2(10) = 6{,}166{,}988{,}769$. Subset-sum reachability is computed with a *packed bitset*: an arbitrary-precision integer R with bit k set iff some subset sums to k; starting from $R = 1$, each atom a updates $R \mid = (R \ll a)$. The final R has $\approx 6.17 \times 10^9$ bits ($\approx 771\,\mathrm{MB}$). One then checks the contiguous segment over the central block [1,027,831,462, 5,139,157,307] is entirely set: 0 integers missing. This was independently re-verified by multiple from-scratch implementations.

The range $10 \leq N \leq 299$ for $Y_0 \leq \min\{\beta, \beta'\}$. For each N, $Y_0(N)$ is computed exactly by the polynomial Dijkstra over $\mathbb{Z}_q$ of §5 (state space $\mathbb{Z}_q$, $q = p_{N+1} \leq p_{300} = 1{,}987$), and $Y_0(N) \leq \min\{\beta(N), \beta'(N)\}$ checked with exact integers. All 290 values pass; the verification is tightest at $N = 10$. Representative normalized data are collected in Appendix A.

Feed residue-completeness for $10 \leq N \leq 13$ (Fact 4.4). Olson's hypothesis holds for all $N \geq 14$, so only the four cases $N = 10, 11, 12, 13$ — where the proof needs residue-completeness but Olson does not apply — require computation. For each, the subset sums $L^1(N)$ are reduced mod $q = p_{N+1} \in \{31, 37, 41, 43\}$ (a $\mathbb{Z}_q$ subset-sum reachability, or an explicit residue-by-residue certificate) to confirm all q residues occur; the largest is $N = 13$, $q = 43$, a 13×43 residue-DP over $\mathbb{Z}_{43}$.

The rational sliver (Theorem 7.1, Step 4). The same packed bitset of $L^2(10)$ is gap-free not only on its central block but downward to $L := 740{,}082{,}854 = 0.11439\ldots \cdot P_{10}$: every integer of $[L, \sigma^2(10)/6]$ is a subset sum of $D_2(10)$. Hence every a/b with $a/b \in [0.1144, B_{10}/6)$ has $aP_{10}/b \in L^2(10)$ (4a). The residual region $a/b \in [B_{N_b}/6, 0.1144)$ forces $N_b \leq 6$, i.e. $b \mid 2{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}13 = 30{,}030$ and $a < 0.1144\, b \leq 3434$; an exhaustive search yields exactly 47 coprime pairs (a, b), each with $aP_N/b \in L^2(N)$ for some $N \in [N_b, 10]$ (4b). The 47-pair list is reproducible by the same reachability routine.

The uniform base γ_{10} (Theorem 7.4). The uniform threshold is seeded at the gap-free floor $\gamma_{10} = 0.11439\ldots$ of $L^2(10)$ — the *same* $N_0 = 10$ packed bitset as the integer base case (Proposition 3.2), reread for its largest hole below the midpoint; no separate large computation is needed. The recurrence's middle sum $\sum_{11}^{300} w_k/p_k = 0.0475$ is evaluated exactly from $w_k = Y_0(k-1)/P_{k-1}$ (the same $\mathbb{Z}_q$ data as Appendix A; the products $w_k p_k$ fall from 9.14 at $k = 13$), and everything past $k = 300$ is bounded analytically by $\sum_{k>300} 4/p_k^{3/2} = 0.0191$ in the proof of Theorem 7.4 — so no w_k beyond $k = 300$ is computed.

Each of these computations is elementary — subset-sum reachability by a packed bitset for the base case and the sliver, a polynomial $\mathbb{Z}_q$ minimum-weight knapsack for Y_0 and the w_k, and direct residue enumeration for the small-N feed — and was independently re-verified. The residue-side computations are polynomial ($O(p_k k)$, not exponential); the only heavy step is the single value-side bitset at $N = 10$ (≈ 771 MB), which serves both the integer base case and the uniform threshold. We record representative numerical data in Appendix A and Table 3; the computations are reproduced by the standard-library scripts of Appendix B, included as ancillary files.

7. The rational case above the threshold

Theorem 1.2 is the integer ($b = 1$) case. The very same engine — the reduction ($\star$), the central block (Lemma 3.1), and the overlap (Proposition 3.4) — already settles *every* rational a/b with squarefree b that lies above an explicit threshold. This upgrades the reach from "integers" to a clean dichotomy (§9).

Throughout this section b is squarefree, N_b denotes the index of the largest prime factor of b (so $p_{N_b} \mid b$ and $p_{N_b+1} \nmid b$), and $M_b := \max(10, N_b)$. "$\omega = 2$ representable" means a finite sum of distinct unit fractions with semiprime denominators.

Theorem 7.1 (above the threshold)**.** *For every squarefree b and every rational $a/b \geq B_{N_b}/6$, the number a/b is $\omega = 2$ representable. In particular every $a/2$ is representable.*

Proof. Set $M_b = \max(10, N_b)$. We prove representability first for $a/b \geq B_{M_b}/6$, then discharge the sliver $[B_{N_b}/6, B_{M_b}/6)$.

Step 1 (the blocks cover $[B_{M_b}/6, \infty)$). By Proposition 3.4, $B_{N+1} \leq 5B_N$ for all $N \geq 10$, so the closed intervals $J_N := [B_N/6, 5B_N/6]$ satisfy $B_{N+1}/6 \leq 5B_N/6$, i.e. J_N and J_{N+1} overlap. Since $B_N \uparrow \infty$, $\bigcup_{N \geq M_b} J_N = [B_{M_b}/6, \infty)$. We also record the sharper ratio used in Theorem 7.4: from $B_{N+1} - B_N = A_N/p_{N+1}$ and $A_N/B_N = 2A_N/(A_N^2 - S_N)$ with $S_N = \sum_{i \leq N} p_i^{-2}$ — decreasing in N, since $x \mapsto 2x/(x^2 - S_N)$ has negative derivative — one gets $A_N/B_N \leq A_{10}/B_{10} < 1.61$, whence

$$\frac{B_{N+1}}{B_N} = 1 + \frac{A_N/B_N}{p_{N+1}} \leq 1 + \frac{1.61}{31} < 1.06 \qquad (N \geq 10). \tag{1}$$

Step 2 (integrality; needs squarefree b). As b is squarefree it is a product of distinct primes, all $\leq p_{N_b}$, so $b \mid P_{N_b} \mid P_N$ for every $N \geq N_b$; hence $aP_N/b = a(P_N/b) \in \mathbb{Z}$. (Squarefreeness is necessary: a square factor $p^2 \mid b$ gives $b \nmid P_N$.)

Step 3 (membership above threshold). Let $a/b \geq B_{M_b}/6$. By Step 1 choose $N \geq M_b$ (≥ 10) with $a/b \in J_N$. Multiplying by P_N and using $\sigma^2(N) = B_N P_N$, the integer aP_N/b (Step 2) lies in $[\sigma^2(N)/6, 5\sigma^2(N)/6]$, hence in $[\lceil \sigma^2(N)/6 \rceil, \lfloor 5\sigma^2(N)/6 \rfloor]$, so $aP_N/b \in L^2(N)$ by Lemma 3.1. By ($\star$) (Proposition 2.3), a/b is $\omega = 2$ representable.

Step 4 (the sliver, only when $N_b < 10$). The interval $[B_{N_b}/6, B_{M_b}/6)$ is nonempty only for $N_b < 10$ (else $M_b = N_b$ and Step 3 suffices). It is discharged by two finite computations (Section 6): (4a) the packed bitset $L^2(10)$ is gap-free not merely on its central block but downward to $0.11439\,P_{10}$, which covers the *entire* sliver of every b with $N_b \geq 7$ (since $B_7/6 > 0.1144$); and (4b) the residual region ($b \mid 30030$, $N_b \leq 6$) is a finite set of exactly 47 coprime pairs (a, b), each verified representable. Together (4a) and (4b) cover $[B_{N_b}/6, B_{M_b}/6)$. □

N	p_N	B_N	$B_N/6$	N	p_N	B_N	$B_N/6$
1	2	0.0000	0.0000	10	29	0.9532	0.1589
2	3	0.1667	0.0278	11	31	1.0027	0.1671
3	5	0.3333	0.0556	12	37	1.0450	0.1742
4	7	0.4810	0.0802	13	41	1.0838	0.1806
5	11	0.5879	0.0980	14	43	1.1215	0.1869
6	13	0.6854	0.1142	15	47	1.1564	0.1927
7	17	0.7644	0.1274	16	53	1.1877	0.1980
8	19	0.8382	0.1397	17	59	1.2162	0.2027
9	23	0.9015	0.1503	18	61	1.2440	0.2073

TABLE 2. The normalized block floor $B_N = \sigma^2(N)/P_N$ and the threshold $B_N/6$ of Theorem 7.1: a squarefree denominator b whose largest prime factor is p_N (so $N_b = N$) is representable for every $a/b \geq B_N/6$. Values are exact rationals shown to four places. From $N = 17$ onward $B_N/6 > \frac{1}{5}$, so the b-uniform bound of Theorem 7.4 is the smaller threshold.

Remark 7.2. For $N_b \geq 10$ one has $M_b = N_b$, so there is no sliver and the threshold is exactly $B_{N_b}/6$, the left endpoint of the lowest block J_{N_b} entering the covering; below it the central-block method provides no guarantee (§9). Table 2 lists B_N and the resulting threshold $B_N/6$. For $b = 2$ one has $N_2 = 1$, $M_2 = 10$, threshold $B_{10}/6 = 0.1589$, and concretely $\frac{1}{2} = \frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{26}+\frac{1}{35}+\frac{1}{39}+\frac{1}{65}+\frac{1}{91}$.

7.1. **A b-uniform threshold.** The threshold $B_{N_b}/6$ grows with the denominator,

$$B_{N_b} \sim \tfrac{1}{2}(\log\log p_{N_b})^2.$$

Reusing the onset radius Y_0 of Section 5 we obtain an *unconditional, b-uniform* threshold. The splitting recursion (S), reindexed $N \to N-1$, reads $L^2(N) = p_N\, L^2(N-1) + L^1(N-1)$. Let γ_N be the *gap-free floor* of $L^2(N)$ — the least γ for which every integer from γP_N up to $\sigma^2(N) - \gamma P_N$ lies in $L^2(N)$, i.e. how far below the central block $L^2(N)$ stays gap-free — and put

$$w_N := \max_{r \in \mathbb{Z}_{p_N}} \min\Big\{\sum_{i\in T} \tfrac{1}{p_i} : T \subseteq \{1,\dots,N-1\},\ \sum_{i\in T} p_i^{-1} \equiv r \pmod{p_N}\Big\}. \tag{2}$$

By Definition 5.1, w_N is *exactly* the normalized onset radius of the feed one level down: $w_N = Y_0(N-1)/P_{N-1}$.

Lemma 7.3 (floor recurrence)**.** *If the subset sums of $\{p_i^{-1} \bmod p_N : 1 \leq i < N\}$ cover $\mathbb{Z}_{p_N}$ (so $w_N < \infty$; Fact 4.4) and $\gamma_{N-1} \leq 0.45\, B_{N-1}$, then $\gamma_N \leq \gamma_{N-1} + w_N/p_N$.*

Proof. By the complement symmetry $m \in L^2(N) \Leftrightarrow \sigma^2(N) - m \in L^2(N)$ it suffices to place every integer $m \in [(\gamma_{N-1} + w_N/p_N)P_N,\ \sigma^2(N)/2]$ in $L^2(N)$. Choose T with $\sum_{i\in T} p_i^{-1} \equiv m\, P_{N-1}^{-1} \pmod{p_N}$ and $\sum_{i\in T} 1/p_i \leq w_N$; then $y := \sum_{i\in T} P_{N-1}/p_i \leq w_N P_{N-1}$ satisfies $y \equiv m \pmod{p_N}$, so $x := (m-y)/p_N \in \mathbb{Z}$. Now $x \geq (m - w_N P_{N-1})/p_N \geq \gamma_{N-1} P_{N-1}$, and $x \leq m/p_N \leq \sigma^2(N)/(2p_N) = \frac{1}{2} B_N P_{N-1} \leq (B_{N-1} - \gamma_{N-1})P_{N-1} = \sigma^2(N-1) - \gamma_{N-1}P_{N-1}$, the last step because $B_{N-1} - \frac{1}{2}B_N \geq B_{N-1}(1 - \frac{1.06}{2}) = 0.47\, B_{N-1} \geq \gamma_{N-1}$ by (1) ($B_N \leq 1.06\, B_{N-1}$, valid for $N \geq 10$) and the hypothesis $\gamma_{N-1} \leq 0.45\, B_{N-1}$. Thus x lies in the gap-free range of $L^2(N-1)$, so $x \in L^2(N-1)$, and $m = p_N x + y \in L^2(N)$ by (S). □

Theorem 7.4 (uniform threshold)**.** *There is an explicit absolute constant $\gamma_\infty \leq \frac{1}{5}$ such that every a/b with b squarefree and $a/b \geq \gamma_\infty$ is $\omega = 2$ representable. Combined with Theorem 7.1, a/b is representable whenever $a/b \geq \min\{B_{N_b}/6,\ \frac{1}{5}\}$, uniformly in b; this improves the b-dependent threshold exactly when b has a prime factor $\geq p_{17} = 59$.*

Proof. For each N the subset sums of $\{p_i^{-1} \bmod p_N : i < N\}$ cover $\mathbb{Z}_{p_N}$ (Fact 4.4). By the tail estimate of Section 5, the $c_0 \approx 2\sqrt{p_N}$ smallest feed atoms already exhaust the residues and each exceeds $p_N/2$, so $w_N = Y_0(N-1)/P_{N-1} \le 2c_0/p_N \le 4/\sqrt{p_N}$; hence $\sum_N w_N/p_N$ converges. Iterating Lemma 7.3 from the base $\gamma_{10} = 0.11439\ldots$ — the gap-free floor of $L^2(10)$, which is the $N_0 = 10$ base case of Theorem 1.2 already in hand (no separate computation; Section 6) —

$$\gamma_N \le \gamma_{10} + \sum_{k=11}^{N} \frac{w_k}{p_k} \le \underbrace{0.1144}_{\gamma_{10}} + \underbrace{0.0475}_{\sum_{11}^{300} w_k/p_k} + \underbrace{0.0191}_{\sum_{k>300} 4p_k^{-3/2}} = 0.1810 < \tfrac{1}{5},$$

the middle sum $\sum_{11}^{300} w_k/p_k = 0.0475$ evaluated exactly from $w_k = Y_0(k-1)/P_{k-1}$ (an $O(p_k k)$ minimum-weight subset computation over $\mathbb{Z}_{p_k}$; the products $w_k p_k$ decrease from 9.1 at $k = 13$), and the tail $\sum_{k>300} 4/p_k^{3/2} = 0.0191$ a convergent prime sum bounded by the covering estimate $w_k \le 4/\sqrt{p_k}$ (Theorem 4.5, valid for $k > k_0 = 120$). Thus $\gamma_N \le 0.181 < \frac{1}{5}$ for all $N \ge 10$; we keep $\frac{1}{5}$ as the (rounder) stated threshold. The iteration runs over $k \ge 11$, where the side condition of Lemma 7.3 is met at every step: $\gamma_{k-1} \le 0.181 \le 0.45\, B_{k-1}$, since $B_{k-1} \ge B_{10} = 0.9532$ gives $0.45\, B_{k-1} \ge 0.429$. Given $a/b \ge \gamma_\infty$ choose $N \ge N_b$ with $B_N \ge a/b + \gamma_\infty$ (possible as $B_N \to \infty$); then $aP_N/b \in \mathbb{Z}$ (Theorem 7.1, Step 2) lies in $[\gamma_N P_N, \sigma^2(N) - \gamma_N P_N] \subseteq L^2(N)$, so a/b is representable by $(\star)$. Finally $B_{N_b}/6 > \frac{1}{5}$ exactly when $N_b \ge 17$ (as $B_{16} = 1.188 < 1.2 < 1.216 = B_{17}$), i.e. when b has a prime factor $\ge p_{17} = 59$. □

8. The rational $\omega = 3$ theorem

The threshold theorem of §7 settles only part of the $\omega = 2$ problem. It nonetheless settles the rational $\omega = 3$ problem of Erdős and Graham in full: every positive rational with squarefree denominator is a finite sum of distinct unit fractions with *sphenic* denominators (three distinct prime factors). The mechanism is a lift that sends a target to a slightly larger one, represents it in $\omega = 2$, and divides by one extra prime; because the lift can reach arbitrarily small targets, it never enters the open core (AC). It rests on the floor-recurrence machinery of §7 — unconditional, via the robustness Lemma 8.4 below — and *not* on the value of the $\omega = 2$ uniform constant, so it is unaffected by the choice of that constant's base. Of its truth the three authors of BEG15 record, in their closing footnote, that one believed it, another doubted it, and the third — "having looked in The BOOK at the answer" — stayed silent ([2]); to our knowledge no proof has appeared.

Theorem 8.1 (rational $\omega = 3$)**.** *Every positive rational a/b with b squarefree is a finite sum of distinct unit fractions, each denominator a product of three distinct primes.*

8.1. **Complementation and the central block.** For $k \ge 1$ write

$$B_k(N) = \sum_{i_1 < \cdots < i_k \le N} 1/(p_{i_1} \cdots p_{i_k}),$$

so $B_2(N) = B_N$. Clearing $1/(p_a p_b p_c)$ by P_N turns it into $P_N/(p_a p_b p_c)$, the product of the *other* $N-3$ primes; hence a/b is $\omega = 3$ representable using primes $\le p_N$ iff $(a/b)P_N$ is a subset sum of the triple-prime atoms

$$\{\, P_N/(p_i p_j p_k) : \; i < j < k \le N \,\},$$

and BEG15's central-block lemma applies.

Theorem 8.2 (BEG15, Lemma 1)**.** *For $N \ge 8$, every integer of $[\frac{1}{6}\sigma^3(N), \frac{5}{6}\sigma^3(N)]$ is such a subset sum, where $\sigma^3(N) = B_3(N)P_N$. Equivalently every a/b with $b \mid P_N$ and $a/b \in [B_3(N)/6,\, 5B_3(N)/6]$ is $\omega = 3$ representable.*

This is BEG15's *published* result (their covering uses Olson's theorem [8]); we cite it and do not reprove it. A concrete instance, from BEG15: with $N = 6$, $\frac{1}{7}P_6 = 4290 \in L_6(3)$, giving

$$\tfrac{1}{7} = \tfrac{1}{30} + \tfrac{1}{42} + \tfrac{1}{66} + \tfrac{1}{70} + \tfrac{1}{78} + \tfrac{1}{105} + \tfrac{1}{110} + \tfrac{1}{154} + \tfrac{1}{165} + \tfrac{1}{195} + \tfrac{1}{273} + \tfrac{1}{286},$$

each denominator a product of three distinct primes (e.g. $30 = 2 \cdot 3 \cdot 5$, $286 = 2 \cdot 11 \cdot 13$). The $\omega = 3$ blocks overlap exactly as the $\omega = 2$ blocks do (Proposition 3.4): the increment is $B_3(N{+}1) - B_3(N) = B_2(N)/p_{N+1}$ (the new triples are pairs joined with p_{N+1}), while every pair $\{i, j\}$ extends to a triple by any third prime $k \notin \{i, j\}$, each triple arising from three pairs, so

$$B_3(N) \;\geq\; \tfrac{1}{3}\,B_2(N)\Big(A_N - \tfrac{1}{2} - \tfrac{1}{3}\Big) \;=\; \tfrac{1}{3}\,B_2(N)\big(A_N - \tfrac{5}{6}\big).$$

Hence for $N \geq 8$ (where $p_{N+1} \geq p_9 = 23$ and $A_N \geq A_8 = 1.4554\ldots$),

$$\frac{B_3(N{+}1)}{B_3(N)} \;\leq\; 1 + \frac{3}{p_{N+1}\,(A_N - \frac{5}{6})} \;\leq\; 1 + \frac{3}{23 \cdot 0.62} \;<\; 1.22 \;<\; 5,$$

so $B_3(N{+}1) \leq 5B_3(N)$, consecutive central blocks $[B_3(N)/6, 5B_3(N)/6]$ meet, and the union over $N \geq \max(8, N_b)$ gives

$$\big[\,B_3(\max(8, N_b))/6,\ \infty\big) \;\subseteq\; \{\,a/b :\ \omega = 3 \text{ representable}\,\}. \tag{3}$$

In particular every $a/b \geq B_3(8)/6 = 0.04174\ldots$ is covered. What BEG15 left open — "recovering very small rational numbers", where "the structure of $L_{n+3}(n)$ is not well understood at the ends of the interval" ([2]) — is the *deep* part $a/b < B_3(N_b)/6$. We settle it by a *lift* to the unconditional $\omega = 2$ theorem.

8.2. **The lift.**

Lemma 8.3 (lift)**.** *Let $t = a/b > 0$ and let r be a prime with $r \nmid b$. If $v := t\,r$ is $\omega = 2$ representable using a prime set Π with $r \notin \Pi$, say $v = \sum_{(i,j)} 1/(q_i q_j)$ with $q_i, q_j \in \Pi$, then $t = \sum_{(i,j)} 1/(q_i q_j r)$ is a sum of distinct sphenic unit fractions.*

Proof. $t = v/r = \sum 1/(q_i q_j r)$. The three primes q_i, q_j, r are distinct because $r \notin \Pi$, and distinct pairs give distinct denominators. Choosing $\Pi \cup \{r\} \subseteq \{p_1, \ldots, p_M\}$ with $b \mid P_M$ makes every term integral over b (then $v \cdot \prod \Pi = aP_M/b \in \mathbb{Z}$ as $\gcd(a, b) = 1$). □

Thus t is covered once $v = tr$ lies in the gap-free range of an $\omega = 2$ cover that omits r. Write $\Pi(M, r) = \{p_1, \ldots, p_M\} \setminus \{r\}$, $B_\Pi(M, r) = \sum_{i<j:\, p_i, p_j \in \Pi} 1/(p_i p_j)$, and let $\gamma_{\mathrm{ex}}(M, r)$ be the $\omega = 2$ gap-free floor of the cover over $\Pi(M, r)$ — the analogue of γ_N (§7) with one ruler prime deleted. By the mechanism of Theorem 7.4 the cover is gap-free on $[\gamma_{\mathrm{ex}}(M, r)\,P_\Pi,\ \sigma_\Pi - \gamma_{\mathrm{ex}}(M, r)\,P_\Pi]$, where $P_\Pi = P_M/r$ and $\sigma_\Pi = B_\Pi(M, r)\,P_\Pi$ is the total of the Π-atoms, so the lift covers

$$t \in \Big[\ \tfrac{\gamma_{\mathrm{ex}}(M,r)}{r},\ \tfrac{B_\Pi(M,r) - \gamma_{\mathrm{ex}}(M,r)}{r}\ \Big] =: \big[\,\mathcal{L}(M, r),\ \mathcal{R}(M, r)\,\big].$$

The lower end $\mathcal{L}(M, r)$ shrinks as r grows, and the reach $\mathcal{R}(M, r) \to \infty$ as $M \to \infty$; once $\gamma_{\mathrm{ex}}(M, r)$ is bounded *uniformly in M and r* (Lemma 8.4 below), every target $t > 0$ is caught by choosing r large enough that $\mathcal{L} < t$ and then M large enough that $\mathcal{R} \geq t$. The order of the two choices is essential: r is fixed *before* $M \to \infty$. Tying r to M instead — say $r = p_{M+1}$ — defeats the argument, since then $\mathcal{R} = (B_M - \gamma_M)/p_{M+1} \to 0$: the numerator grows only like $\frac{1}{2}(\log\log p_M)^2$ while $p_{M+1} \to \infty$, so that coupling reaches only ever-smaller targets near 0, never a fixed $t > 0$.

8.3. **Floor-robustness: deleting one ruler prime.** The one genuinely new estimate is that deleting a prime from the $\omega = 2$ ruler raises the floor by a bounded amount, uniformly in the level M and in which prime is deleted. Recall the covering cost (2) of §7 driving $\gamma_k \le \gamma_{k-1} + w_k/p_k$. Deleting ruler prime p_s ($s < k$) gives the *excluded cost*

$$w_k^{\mathrm{ex}}(s) = \max_{t \in \mathbb{Z}_{p_k}} \min\Big\{ \sum_{i \in T} \tfrac{1}{p_i} : T \subseteq \{1, \dots, k-1\} \setminus \{s\},\ \sum_{i \in T} p_i^{-1} \equiv t \pmod{p_k} \Big\}, \tag{4}$$

and the recurrence $\gamma_{\mathrm{ex}}(k, r) \le \gamma_{\mathrm{ex}}(k-1, r) + w_k^{\mathrm{ex}}(s)/p_k$, valid whenever the punctured ruler still covers $\mathbb{Z}_{p_k}$ (so $w_k^{\mathrm{ex}} < \infty$).

Lemma 8.4 (floor-robustness, uniform in r and M)**.** *For every prime r and every $M \ge 11$, $\gamma_{\mathrm{ex}}(M, r) \le 0.20$.*

Proof. Work at level $M \ge 11$ (free, since $b \mid P_{10} \Rightarrow b \mid P_{11}$); this matters because the $M = 10$ small-prime floors are large ($\gamma_{\mathrm{ex}}(10, 5) = 0.165$) yet all drop by $M = 11$. Bound $\gamma_{\mathrm{ex}}(M, r) \le \text{base} + \sum_{k=12}^{M} w_k^{\mathrm{ex}}/p_k$ piecewise.

Base. Exactly (packed bitset; Table 3), $\max_{r \le p_{11}} \gamma_{\mathrm{ex}}(11, r)$ is attained at $r = 19$, where the exact floor is $\gamma_{\mathrm{ex}}(11, 19) = 0.11602\ldots$ (the table displays four decimals; we carry the exact value, not a truncation); for $r \ge p_{12}$ the base is the full-ruler $\gamma_{11} = 0.0957$. So base ≤ 0.11602. (The floor is *non-monotone* in r, so the small primes $r = 2, 3, 5, 7$ must be — and are — included in this maximum.)

Exact cost, $12 \le k \le 220$. With $w_k^{\max} := \max_{1 \le s < k} w_k^{\mathrm{ex}}(s)$ the worst single-prime deletion, $\sum_{k=12}^{220} w_k^{\max}/p_k = 0.0498$ (worst single level $w_k p_k = 14.32$ at $k = 12$), and the punctured ruler was verified to cover $\mathbb{Z}_{p_k}$ for *every* deletion $s < k$ at every such k (zero failures).

Tail, $k > 220$. The deletion crossover is $k_0 = 120$: for $k \ge 120$ the $\lceil \sqrt{4p_k - 3} \rceil + 1$ largest ruler primes all exceed $p_k/2$, so after one deletion at least $\sqrt{4p_k - 3}$ of them remain $> p_k/2$, each of inverse-weight $< 2/p_k$; by Theorem 4.5 their subset sums already exhaust $\mathbb{Z}_{p_k}$, whence $w_k^{\mathrm{ex}} \le \sqrt{4p_k - 3} \cdot 2/p_k < 4/\sqrt{p_k}$. As $220 > k_0$,

$$\sum_{k > 220} w_k^{\mathrm{ex}}/p_k \ \le \ \sum_{k > 220} \frac{4}{p_k^{3/2}} \ = \ 0.024$$

(a convergent prime sum, evaluated to convergence). The recurrence's side condition (the analogue of Lemma 7.3's) holds at every step: it requires $\gamma_{\mathrm{ex}}(k-1, r) \le B_\Pi(k-1, r) - B_\Pi(k, r)/2$. Deleting r removes mass $(A_M - 1/r)/r \le (A_M - \frac{1}{2})/2$ from B_M (the removed mass is maximal at $r = 2$, as its r-derivative is negative for $A_M r > 2$), so $B_\Pi(k-1, r) \ge B_{11} - (A_{11} - \frac{1}{2})/2 > 1.002 - 0.533 = 0.46$ for *every* prime r and $k \ge 12$; and the level increment is $B_\Pi(k, r) - B_\Pi(k-1, r) \le A_{k-1}/p_k \le A_{11}/p_{12} = 1.566/37 < 0.043$ for all $k \ge 12$ (the quotient A_{k-1}/p_k is largest at $k = 12$, since p_k grows faster than A_{k-1}). Hence

$$\begin{aligned} B_\Pi(k-1, r) - \tfrac{1}{2} B_\Pi(k, r) &= \tfrac{1}{2} B_\Pi(k-1, r) - \tfrac{1}{2}\big(B_\Pi(k, r) - B_\Pi(k-1, r)\big) \\ &\ge \tfrac{0.46}{2} - \tfrac{0.043}{2} = 0.208 > 0.1899 \ge \gamma_{\mathrm{ex}}(k-1, r), \end{aligned}$$

inductively (using the assembled bound $\gamma_{\mathrm{ex}} \le 0.1899$ just below). Summing, $\gamma_{\mathrm{ex}}(M, r) \le 0.11602 + 0.04980 + 0.02407 = 0.1899 < 0.20$ for every prime r and every $M \ge 11$ (using the *exact* base 0.11602 and the converged tail ≤ 0.02407; the round bound 0.20 leaves ample margin, and all downstream uses need only $\gamma_{\mathrm{ex}} < 0.298$). □

Remark 8.5. The bound is uniform over *which* prime is deleted because it uses the worst-case cost $w_k^{\max}$, which dominates every single-r value; the actual per-r floors are smaller ($\gamma_{\mathrm{ex}}(M, r) \le 0.160$ for all r, worst at $r = 19$; Table 3). Only the uniform bound $\gamma_{\mathrm{ex}} \le 0.20$ (assembled value 0.1899) is load-bearing.

8.4. **Proof of the theorem.** With Lemma 8.4 in hand, no interval-chaining is needed: the lift prime can simply be chosen *large enough for the target*.

Proof of Theorem 8.1. Fix the target $t = a/b > 0$ in lowest terms, b squarefree. Choose a prime r with

$$r \nmid b, \qquad r \geq 11, \qquad r > 0.20/t$$

(only finitely many primes divide b, so such r exists). Then choose a level $M \geq \max(N_b, \pi(r), 11)$, where π is the prime-counting function (so $r = p_{\pi(r)}$ and $M \geq \pi(r)$ places r among $p_1, \ldots, p_M$), large enough that

$$B_\Pi(M, r) \;\geq\; t\,r + 0.20,$$

which is possible since $B_\Pi(M, r) \to \infty$ as $M \to \infty$ (adding primes adds positive pair-reciprocals, and $B_\Pi(M, r) \geq B_M - (A_M - 1/r)/r \to \infty$). Put $v := t\,r$. By the choice of r,

$$\gamma_{\mathrm{ex}}(M, r) \;\leq\; 0.20 \;<\; v \;\leq\; B_\Pi(M, r) - 0.20 \;\leq\; B_\Pi(M, r) - \gamma_{\mathrm{ex}}(M, r)$$

(Lemma 8.4 for both ends), so v lies in the gap-free band of the $\omega = 2$ cover over $\Pi = \Pi(M, r)$. Moreover $v \cdot P_\Pi = (ar/b)(P_M/r) = aP_M/b \in \mathbb{Z}$, since b is squarefree with all prime factors among $\{p_1, \ldots, p_M\}$ ($M \geq N_b$) and $r \nmid b$. Hence v is $\omega = 2$ representable over Π, which omits r, and Lemma 8.3 turns this into a sphenic representation $t = \sum 1/(q_i q_j r)$. □

r	2	3	5	7	11	13	17	19	23	29	31
$\gamma_{\mathrm{ex}}(11, r)$	.0866	.0938	.1044	.1086	.1047	.1063	.1111	**.1160**	.1088	.1081	.1144

TABLE 3. Exact base floors at $M = 11$ (packed bitset; calibrated against Theorem 7.4, whose full-ruler run reproduces $\sum_{13}^{2000} w_k/p_k = 0.03388$, $w_{13}p_{13} = 9.14$). Entries are rounded to four decimals; the maximum, at $r = 19$, is exactly $\gamma_{\mathrm{ex}}(11, 19) = 0.11602\ldots$. Assembled uniform bound $\gamma_{\mathrm{ex}}(M, r) \leq 0.11602 + 0.04980 + 0.02407 = 0.1899 < 0.20$ for all primes r, all $M \geq 11$ (worst single level $w_k p_k = 14.32$ at $k = 12$; deletion crossover $k_0 = 120$). Regime-II reach e.g. $\mathcal{R}(10, 11) = (0.822 - 0.14605)/11 = 0.0615$, $\mathcal{R}(10, 13) = (0.841 - 0.15042)/13 = 0.0531$.

8.5. **Closure: all $\omega \geq 3$.** A single descent upgrades Theorem 8.1 to the whole hierarchy. We position this as a corollary: it is conceptually routine — each step trades one ω for one more prime, and higher ω only helps (more atoms, larger margins) — the sole technicality being that the induction must avoid a growing finite set of primes, which costs nothing because those primes are ours to choose (below). This parallels BEG15's one-line "similar arguments" treatment of the $\omega \geq 4$ integer case.

Corollary 8.6 ($\omega \geq 3$ closure)**.** *For every integer $k \geq 3$, every positive rational a/b with b squarefree is a finite sum of distinct unit fractions $1/m$, each m squarefree with exactly k distinct prime factors.*

We induct on a strengthened statement carrying a finite avoidance set of *large* primes. For $t \geq 0$ let $K(t)$ be the least $K \geq 220$ such that for every $k > K$, with $m_k := \lceil \sqrt{4p_k - 3}\,\rceil$,

$$\text{(i) } k - 1 - m_k \;\geq\; t + 1, \qquad \text{(ii) } p_{\,k-m_k-t-1} \;>\; p_k/2.$$

$K(t)$ is finite for every t and nondecreasing in t: the number of ruler primes exceeding $p_k/2$ is $k - \pi(p_k/2)$, which by the elementary Chebyshev bounds grows linearly in k, while $m_k + t + 1 = O(\sqrt{k \log k}) + t$ grows slower. (For $t = 1$, condition (ii) is exactly the deletion crossover of Lemma 8.4,

which holds from $k_0 = 120$ on; the floor 220 in the definition is for convenience, aligning with the exactly computed range.) The induction statement is

(H_k) : for every squarefree-denominator a/b and every finite set T of primes with $T \cap \text{primes}(b) = \varnothing$ and $\min(T) > p_{K(|T|+k)}$, the rational a/b is a finite sum of distinct $\omega = k$ unit fractions whose denominators use no prime in T.

(The avoidance set lets each descent step forbid one more prime; the largeness condition costs nothing, because every prime ever placed in T is chosen by us, and it is vacuous for $T = \varnothing$.) Write $\gamma_2(T)$ for the $\omega = 2$ gap-free floor with the primes of T deleted from the ruler, P'_N for the product of the first N primes not in T, and $B_2(N;T) = \sigma^2(N;T)/P'_N$.

Lemma 8.7 (finite-set floor-robustness, uniform)**.** *Let T be a finite set of primes with $\min(T) > p_{K(|T|)}$. Then $\gamma_2(N;T) \le 0.181 + 0.024 = 0.205$ for all $N > K(|T|)$. Hence the gap-free band of $L^2(N;T)$ is nonempty for all such N, and every squarefree-b rational in the corresponding window with $\text{primes}(b) \cap T = \varnothing$ is $\omega = 2$ representable using primes $\notin T$.*

Proof. Write $t = |T|$, $K = K(t)$. *Levels $k \le K+1$ are untouched:* the level-k ruler consists of the first $k-1$ primes, all $\le p_K < \min(T)$, so through level $K+1$ the T-avoiding system *coincides* with the full system and Theorem 7.4 gives $\gamma_2(K+1;T) = \gamma_{K+1} \le 0.181$. *Levels $k > K+1$:* at most t ruler primes are deleted. Among the $m_k + t + 1$ largest ruler primes at least $m_k + 1$ survive the deletion; by (ii) all of them exceed $p_k/2$, so they are distinct nonzero residues mod p_k of inverse-weight $< 2/p_k$ each, and by Theorem 4.5 their subset sums exhaust $\mathbb{Z}_{p_k}$. Hence the covering holds at every level and $w_k^T \le m_k \cdot 2/p_k < 4/\sqrt{p_k}$, exactly as in Lemma 8.4. The recurrence's side condition holds throughout: inductively $\gamma_2(k-1;T) \le 0.205$, while $B_2(k-1;T) \ge B_{220} \ge B_{10} > 0.95$ (the first 220 primes are T-free) and the level increment is $B_2(k;T) - B_2(k-1;T) \le A_{k-1}/p_k \le \frac{(k-1)/2}{k-1} = \frac{1}{2}$ (crudely, $A_{k-1} \le (k-1)/2$ and $p_k \ge k-1$), so

$$B_2(k-1;T) - \tfrac{1}{2}B_2(k;T) = \tfrac{1}{2}B_2(k-1;T) - \tfrac{1}{2}\big(B_2(k;T) - B_2(k-1;T)\big) \ge \tfrac{0.95}{2} - \tfrac{1}{4} = 0.225 > 0.205.$$

Summing the recurrence from the base $K+1$,

$$\gamma_2(N;T) \;\le\; 0.181 + \sum_{k>K} \frac{4}{p_k^{3/2}} \;\le\; 0.181 + \sum_{k>220} \frac{4}{p_k^{3/2}} \;=\; 0.181 + 0.024 \;=\; 0.205,$$

uniformly in N and in T. Since $B_2(N;T) \to \infty$, the band is nonempty for all $N > K$, and membership of the integer aP'_N/b in the band gives the representation by the reduction $(\star)$ run over the T-free primes. □

Lemma 8.8 ((H_3))**.** (H_3) *holds.*

Proof. Given the target $t_0 = a/b$ in lowest terms and T with $T \cap \text{primes}(b) = \varnothing$ and $\min(T) > p_{K(|T|+3)}$; write $t = |T|$. Exactly as in the proof of Theorem 8.1, but with Lemma 8.7 in place of Lemma 8.4: choose a prime r with

$$r \notin T \cup \text{primes}(b), \qquad r > p_{K(t+3)}, \qquad r > 0.205/t_0,$$

and set $T' := T \cup \{r\}$, so $|T'| = t+1$ and $\min(T') > p_{K(t+3)} \ge p_{K(t+1)}$ (K is nondecreasing): Lemma 8.7 applies to T' with floor $\gamma_2(M;T') \le 0.205$ at every level $M > K(t+1)$. Choose $M \ge \max(N_b, \pi(r))$, $M > K(t+1)$, with $B_2(M;T') \ge t_0 r + 0.205$ (possible as $B_2(M;T') \to \infty$). Then $v := t_0 r$ satisfies $\gamma_2(M;T') \le 0.205 < v \le B_2(M;T') - \gamma_2(M;T')$, and $v \cdot P'_M = aP'_M r/b \in \mathbb{Z}$ (all prime factors of b lie among the first M primes and outside T'). So v is $\omega = 2$ representable using primes $\notin T'$ — in particular avoiding both T and r — and Lemma 8.3 gives $t_0 = \sum 1/(q_i q_j r)$ with $\omega = 3$ denominators using no prime in T. □

Lemma 8.9 (descent $(H_{k-1}) \Rightarrow (H_k)$, $k \ge 4$)**.** *If (H_{k-1}) holds then so does (H_k).*

Proof. Given a/b (lowest terms, b squarefree) and finite T with $T \cap \text{primes}(b) = \varnothing$ and $\min(T) > p_{K(|T|+k)}$, choose a prime $r > p_{K(|T|+k)}$ with $r \notin T \cup \text{primes}(b)$ (infinitely many exist). Put $v := ar/b$; since $\gcd(ar, b) = 1$, v is in lowest terms with squarefree denominator b, so $\text{primes}(v) = \text{primes}(b)$ and $(T \cup \{r\}) \cap \text{primes}(v) = \varnothing$. Moreover $T \cup \{r\}$ has size $|T| + 1$ and $\min(T \cup \{r\}) > p_{K(|T|+k)} = p_{K((|T|+1)+(k-1))}$, so the hypothesis of (H_{k-1}) is met. By (H_{k-1}) applied to v with forbidden set $T \cup \{r\}$, $v = \sum_i 1/m_i$ with each m_i squarefree, $\omega(m_i) = k - 1$, using no prime in $T \cup \{r\}$, the m_i distinct. Then $a/b = v/r = \sum_i 1/(m_i r)$, and each $m_i r$ is squarefree with $\omega(m_i r) = k$ (as $r \nmid m_i$), uses no prime in T, and the $m_i r$ are distinct. □

Proof of Corollary 8.6. Induction on k: base $k = 3$ is Lemma 8.8, step $k \geq 4$ is Lemma 8.9. Taking $T = \varnothing$ in (H_k) — for which the largeness condition is vacuous — gives the corollary. □

9. Scope and open problems

9.1. **Scope.** For squarefree denominator b, the results of this paper give the complete picture of the Erdős–Graham problem across ω:

ω	coverage of a/b (b squarefree)	status
≥ 3	**every** a/b	this paper (Thm 8.1, Cor. 8.6)
$= 2$	every $a/b \geq \min\{B_{N_b}/6, \frac{1}{5}\}$	this paper (Thms 7.1, 7.4)
	$a/b < \min\{B_{N_b}/6, \frac{1}{5}\}$	**open**: the deep core (AC)

The squarefree hypothesis is intrinsic: a square factor $p^2 \mid b$ makes $b \nmid P_N$, so $(\star)$ cannot place $P_N(a/b)$ as an integer subset sum. The $\omega = 2$ split is a clean dichotomy with no gap: for $N_b \geq 10$ the threshold $B_{N_b}/6$ is exactly the bottom-edge onset, and $\frac{1}{5}$ is the current best *unconditional* value of it (Theorem 7.4), not an endpoint. By $(\star)$ the open part is the placement of $P_N(a/b)$ *below* the central block, in the low range $[\,0, \frac{1}{6}\sigma^2(N)\,)$ of $L^2(N)$ — the obstruction BEG15 isolate already for $\omega = 3$ ("the structure of $L_{n+3}(n)$ is not well understood at the ends of the interval", [2], p. 8; their $a(k)$, with $a(2) = 23$, marks the block start). Since the descent (Corollary 8.6) makes $\omega \geq 4$ no separate result — as in BEG15's one-line "similar arguments" treatment of the $\omega \geq 4$ integer case — the entire remaining content of the problem lives in the single $\omega = 2$ band $a/b < B_{N_b}/6$, where the feed is thinnest.

9.2. **The open core: anti-concentration.** Write $\gamma_N = \inf\{\gamma : [\gamma P_N,\, \sigma^2(N) - \gamma P_N] \cap \mathbb{Z} \subseteq L^2(N)\}$ for the true gap-free floor.

Conjecture 9.1 ((AC), the full close)**.** $\gamma_N \to 0$ as $N \to \infty$.

Proposition 9.2. *If* (AC) *holds then every positive rational a/b (b squarefree) is $\omega = 2$ representable: for fixed a/b the level N is unbounded (as $b \mid P_N$ for all $N \geq N_b$) and $B_N \to \infty$, so $\gamma_N \to 0$ yields some $N \geq N_b$ with $\gamma_N \leq a/b \leq B_N - \gamma_N$; then $(a/b)P_N$ lies in the gap-free band of $L^2(N)$ and the reduction $(\star)$ gives the representation. (This would also remove the lift from §8, giving the full rational $\omega = 3$ statement directly.)*

Appendix A. Numerical data for the exact range $10 \leq N \leq 299$

The induction step requires $Y_0(N) \leq \min\{\beta(N), \beta'(N)\}$ for $10 \leq N \leq 299$ (Theorem 5.4, exact range). Since $Y_0(N)$, $\beta(N)$, $\beta'(N)$ are subset sums of the atoms P_N/p_i and hence have hundreds of digits, we tabulate the *normalized* quantities obtained by dividing by P_N. Thus $y_0(N) := Y_0(N)/P_N$ is the normalized onset radius (a sum of reciprocals of primes), and the normalized strip length is $A_N/6$, since $\beta(N)/P_N$ and $\beta'(N)/P_N$ both lie within $q/P_N < 10^{-7}$ of $A_N/6$ by Lemmas 5.2–5.3. The last column is the margin $\mu(N) := A_N/6 - y_0(N)$, which therefore equals $\min\{\beta(N), \beta'(N)\}/P_N - Y_0(N)/P_N$ up to less than 10^{-7}.

Across the full range $10 \leq N \leq 299$ (all 290 values) the margin is positive; it is *smallest at the seed* $N = 10$, where $\mu(10) = 0.02372$, and stays $\geq 0.02372 > 0$ throughout. The growth is not

strictly monotone — there are a few local dips (e.g. μ falls from 0.15900 at $N = 15$ to 0.15759 at $N = 16$) — but the minimum is attained only at the seed (consistent with Remark 3.3: at the omitted value $N = 9$ one has $Y_0(9) > \beta(9)$, i.e. negative margin). A representative sample follows.

N	$q = p_{N+1}$	$y_0(N) = Y_0(N)/P_N$	$A_N/6$	$\mu(N)$
10	31	0.23186	0.25557	0.02372
11	37	0.23186	0.26095	0.02909
12	41	0.22286	0.26545	0.04259
13	43	0.18716	0.26952	0.08236
14	47	0.15160	0.27339	0.12180
15	53	0.11794	0.27694	0.15900
16	59	0.12249	0.28009	0.15759
17	61	0.10095	0.28291	0.18196
18	67	0.10118	0.28564	0.18446
19	71	0.08368	0.28813	0.20445
20	73	0.07901	0.29048	0.21147
22	83	0.06483	0.29487	0.23004
25	101	0.05198	0.30047	0.24849
30	127	0.04314	0.30830	0.26516
40	179	0.02685	0.31957	0.29271
50	233	0.01838	0.32784	0.30946
75	383	0.00985	0.34185	0.33199
100	547	0.00686	0.35106	0.34419
150	877	0.00409	0.36314	0.35905
200	1229	0.00279	0.37118	0.36839
250	1597	0.00209	0.37712	0.37503
299	1987	0.00167	0.38174	0.38007

(The exact integers underlying the normalized values are reproducible by the $\mathbb{Z}_q$ shortest-path computation of §5.2; for example $Y_0(10) = 1{,}500{,}040{,}080$, $\beta(10) = 1{,}653{,}479{,}725$, with $P_{10} = 6{,}469{,}693{,}230$ recovering $y_0(10) = 0.23186$ and $A_{10}/6 = 0.25557$.)

Appendix B. Reproducibility scripts

The finite computations of §6 are reproduced by three short, self-contained Python scripts (standard library only; exact integer and `Fraction` arithmetic), included as ancillary files:

script	reproduces
`blocks.py`	$\sigma^2(N)$, B_N, $B_3(N)$, $B_\Pi(M,r)$, the floor γ_N, central-block gap-freeness; the deletion floor $\gamma_{\mathrm{ex}}(M,r)$; the 47-pair residual sliver
`onset.py`	$y_0(N) = Y_0(N)/P_N$ vs $A_N/6$ and the exact integers $Y_0(N), \beta(N), \beta'(N)$; feed residue-completeness; the sums $\sum_{11}^{K} w_k/p_k$, $\sum_{12}^{K} w_k^{\max}/p_k$
`olson_tail.py`	the crossover $k_0 = 120$ and the tails $\sum_{k>K_0} 4/p_k^{3/2}$

The value-side reachability (`blocks.py`) is a packed-bitset subset sum; the residue-side maps (`onset.py`, `olson_tail.py`) are polynomial $\mathbb{Z}_q$ knapsacks. The residue-side runs, the sliver enumeration, and the symmetric-function constants are light (seconds); the only heavy steps are the value-side bitsets — the $N = 10$ base case (≈ 771 MB, ≈ 25 s) and the $N = 11$ deletion floors $\gamma_{\mathrm{ex}}(11, r)$ of the $\omega = 3$ base (≈ 5 GB, ≈ 25 s each); for $N \le 9$, `blocks.py` validates the method in seconds. The following commands reproduce every load-bearing number:

```
$ python3 blocks.py 10          # integer base case (~771 MB)
  N=10  B_N=0.95321  gamma_N=0.11439  central block [1027831462,5139157307]
        gap-free: True
$ python3 blocks.py sliver      # the residual rational sliver
```

```
  == residual sliver (Thm 7.1, Step 4b): 47 pairs ==
$ python3 blocks.py omega3      # omega=3 symmetric-function constants
  B_3(8)/6 = 0.04174   B_Pi(10,drop 11) = 0.822   B_Pi(10,drop 13) = 0.841
$ python3 blocks.py ex 11 8     # omega=3 deletion floor (~5 GB)
  gamma_ex(11, p_8=19) = 0.11602
$ python3 onset.py 300          # onset, residue-completeness, w_k sums
  N=10  y0=0.23186  A_N/6=0.25557                      (onset radius)
  Y_0(10)=1500040080  beta(10)=1653479725  Y_0<=min: True
  feed residue-completeness, q in {31,37,41,43}: True  (Olson-fails levels)
  sum_11^300 w_k/p_k     = 0.0475                       (uniform threshold)
  sum_12^220 w_k^max/p_k = 0.0498   worst w_k*p_k=14.32 @ k=12   (omega=3)
$ python3 olson_tail.py         # crossover and analytic tails
  k_0 = 120  (p_120 = 659)
  sum_{k>300} 4/p_k^{3/2} -> 0.0191   sum_{k>220} -> 0.0239
```

Acknowledgements

This paper builds directly on the method of Butler, Erdős and Graham, whose induction skeleton — the splitting recursion, the central-block propagation, and the Olson-controlled feed — it adapts throughout; the debt to their work is pervasive.

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