# SEMIPRIME EGYPTIAN FRACTIONS FOR THE RATIONALS: UNCONDITIONAL PARTIAL RESULTS AND AN ISOLATED OPEN CORE

Abstract. We study which positive rationals $a/b$ are sums of *distinct* unit fractions $1/(p_i p_j)$ with $p_i < p_j$ prime (squarefree $\omega = 2$ semiprime denominators) — the rational, $\omega = 2$ case of a problem of Erdős and Graham (Erdős #306), whose integer case is settled in a companion paper, following Butler–Erdős–Graham for $\omega = 3$. We prove two unconditional results: (A) for every squarefree $b$, every $a/b \geq B_{N_b}/6$ is so representable, with the threshold equal to the exact bottom-edge boundary; and (B) for every squarefree $b$, the representable $a/b$ have natural density 1. As an application we give what is, to our knowledge, the first complete proof of the *rational* $\omega = 3$ statement of Erdős and Graham — every $a/b$ with $b$ squarefree is a sum of distinct *sphenic* unit fractions $1/(p_i p_j p_k)$ — which BEG15 left unpublished and whose truth their authors disagreed on; the proof lifts the unconditional $\omega = 2$ threshold theorem and never touches the open core. We isolate the remaining ($\omega = 2$) content in the *deep* regime $a/b < B_{N_b}/6$ to a single explicit, Hardy–Littlewood–free additive estimate, and prove (C) a conditional deep-almost-all theorem under an explicit two-point hypothesis ($\star_2$). We record that the deep regime is governed by a genuine *anti-concentration* mechanism — not, as a naive reading suggests, by a Hardy–Littlewood prime-pair condition. The hypothesis ($\star_2$) reduces to the (open) randomness of the four-fold additive energy of the prime inverses $\{p^{-1} \bmod q\}$; the full close ($\star$) requires, in addition, the all-order (value-side) anti-concentration of the subset-sum set.

## 1. Introduction

Erdős and Graham [3] asked (Erdős problem #306 [4], Guy D11) whether, for squarefree $b$, every $a/b$ is an Egyptian fraction whose denominators each have exactly $\omega$ distinct prime factors. For $\omega = 3$ the integer case was proved by Butler, Erdős and Graham [2] (*BEG15*); the $\omega = 2$ integer case was left open there as a conjecture and is resolved in our companion paper [1]. Here we study the *rational* $\omega = 2$ problem. In closely related recent work, Czenky, McGovern, Plavnik, Rowell and Watkins [8] study the Egyptian-fraction equation with the prime factors of the denominators constrained to a *fixed finite set* of primes (a smoothness constraint), emphasizing the greedy algorithm and explicit bounds; our constraint is complementary — a fixed *number* ($\omega = 2$) of prime factors drawn from *all* primes.

**Definition 1.** A positive rational $r$ is $\omega = 2$ *representable* if $r = \sum_{(i,j)\in S} \frac{1}{p_i p_j}$ for a finite set $S$ of pairs $i < j$ of distinct prime indices (so the denominators are distinct squarefree semiprimes).

Throughout, $p_i$ is the $i$-th prime, $P_N = \prod_{i\leq N} p_i$, $D_2(N) = \{P_N/(p_i p_j) : 1 \leq i < j \leq N\}$, and $L^2(N) = \{\sum S : S \subseteq D_2(N)\}$ is the set of subset sums. Set

$$B_N = \sum_{1\leq i<j\leq N} \frac{1}{p_i p_j} = \tfrac{1}{2}\big(A_N^2 - S_N\big), \qquad A_N = \sum_{i\leq N} \tfrac{1}{p_i}, \quad S_N = \sum_{i\leq N} \tfrac{1}{p_i^2},$$

so $B_N \sim \frac{1}{2}(\log\log N)^2 \to \infty$, and $\sigma^2(N) = B_N P_N = \max L^2(N)$. For squarefree $b$ let $N_b$ be the index of the largest prime factor of $b$.

The basic reduction (clearing denominators by $P_N$) is:

$$r \text{ is representable using primes} \leq p_N \iff P_N\, r \in L^2(N). \tag{1}$$

Since $B_N \to \infty$ and $N$ is unbounded, *rate of decay plays no role in existence*: a fixed deep target $r = a/b$ is representable as soon as $P_N r \in L^2(N)$ for some $N \geq N_b$ (Remark 6).

*Date*: June 10, 2026.

**Main results.**

- **Theorem 2 (unconditional).** For every squarefree $b$ and every $a/b \geq B_{N_b}/6$, $a/b$ is $\omega = 2$ representable. The threshold $B_{N_b}/6$ is the exact bottom-edge boundary (for $N_b \geq 10$ it coincides with the onset of the open deep regime).
- **Theorem 6 (unconditional).** For every squarefree $b$, the $\omega = 2$ representable $a/b$ have natural density 1 in $\{a/b : \gcd(a, b) = 1\}$.
- **Theorem 7 (conditional on ($\star_2$)).** Under the explicit two-point estimate ($\star_2$) of §5, for every fixed deep scale $\theta > 0$ the proportion of representable $a/b$ at scale $\theta$ tends to 1.
- **Theorem 4 (unconditional, $\omega = 3$).** Every $a/b$ with $b$ squarefree is a sum of distinct sphenic unit fractions $1/(p_i p_j p_k)$ — the full rational $\omega = 3$ statement of Erdős and Graham (BEG15, "never published"; §3). The proof needs only Theorem 3 (lifted, one prime excluded) and BEG15's published central block; it is as secure as Theorem 3.

The paper is *not* a full resolution: $a/b < B_{N_b}/6$ (the deep regime) is open. Our final contribution (§7) is to pin the open core to one explicit, Hardy–Littlewood–free statement ($\star$) and to record why it is a genuine anti-concentration problem.

## 2. The above-threshold theorem

We use the following central-block theorem, proved in our companion paper [1] by a Bleicher–Erdős–Graham-style induction re-seeded at the base $N_0 = 10$ (the $9 \to 10$ seam fails). It is the $\omega = 2$ integer engine.

**Theorem 1** (Central block; companion paper [1]). *For all $N \geq 10$, $L^2(N) \supseteq \big[\lceil \sigma^2(N)/6 \rceil,\ \lfloor 5\sigma^2(N)/6 \rfloor\big] \cap \mathbb{Z}$; that is, $L^2(N)$ contains every integer in the central block $[\sigma^2(N)/6,\ 5\sigma^2(N)/6]$.*

**Theorem 2** (Above-threshold). *For every squarefree $b$ and every rational $a/b \geq B_{N_b}/6$, $a/b$ is $\omega = 2$ representable. In particular (Corollary, $b = 2$) every $a/2$ is representable.*

*Proof.* Set $M_b = \max(10, N_b)$. We first prove representability for $a/b \geq B_{M_b}/6$, then discharge the sliver $[B_{N_b}/6, B_{M_b}/6)$.

*Step 1 (the blocks cover $[B_{M_b}/6, \infty)$).* From $B_{N+1} - B_N = \sum_{i \leq N} \frac{1}{p_i p_{N+1}} = A_N/p_{N+1}$ we get

$$\frac{B_{N+1}}{B_N} = 1 + \frac{A_N/B_N}{p_{N+1}}, \qquad \frac{A_N}{B_N} = \frac{2A_N}{A_N^2 - S_N}.$$

The map $x \mapsto 2x/(x^2 - S_N)$ has derivative $-2(x^2 + S_N)/(x^2 - S_N)^2 < 0$, so $A_N/B_N$ is decreasing in $A_N$, hence in $N$; therefore $A_N/B_N \leq A_{10}/B_{10}$ for all $N \geq 10$. Numerically $A_{10} = 1.53344\ldots$, $S_{10} = 0.44501\ldots$, $B_{10} = (A_{10}^2 - S_{10})/2 = 0.95321\ldots$, so $A_{10}/B_{10} = 1.60871\ldots < 1.61$. With $p_{N+1} \geq p_{11} = 31$ for $N \geq 10$,

$$\frac{B_{N+1}}{B_N} \leq 1 + \frac{1.61}{31} = 1.05194\ldots < 1.06 \qquad (N \geq 10). \tag{2}$$

In particular $B_{N+1} \leq 5B_N$, so the closed intervals ("blocks") $J_N := [B_N/6, 5B_N/6]$ satisfy $B_{N+1}/6 \leq 5B_N/6$, i.e. $J_N$ and $J_{N+1}$ overlap. Since $B_N \to \infty$, $\bigcup_{N \geq M_b} J_N = [B_{M_b}/6, \infty)$.

*Step 2 (integrality).* As $b$ is squarefree, $b \mid \prod_{p \leq p_{N_b}} p = P_{N_b} \mid P_N$ for $N \geq N_b$; hence $aP_N/b = a(P_N/b) \in \mathbb{Z}$. (Squarefreeness is necessary: a square factor $p^2 \mid b$ gives $b \nmid P_N$.)

*Step 3 (membership above threshold).* Let $a/b \geq B_{M_b}/6$. By Step 1 choose $N \geq M_b$ ($\geq 10$) with $a/b \in J_N = [B_N/6, 5B_N/6]$. Multiplying by $P_N$ and using $\sigma^2(N) = B_N P_N$, the integer $aP_N/b$ (Step 2) lies in $[\sigma^2(N)/6,\ 5\sigma^2(N)/6]$, hence in $[\lceil \sigma^2/6 \rceil, \lfloor 5\sigma^2/6 \rfloor]$, so $aP_N/b \in L^2(N)$ by Theorem 1. By (1) a subset $S \subseteq D_2(N)$ with $\sum S = aP_N/b$ yields $a/b = \sum_{(i,j) \in S} 1/(p_i p_j)$ with distinct semiprime denominators.

*Step 4 (sliver discharge, lowering the threshold to $B_{N_b}/6$).* The interval $[B_{N_b}/6, B_{M_b}/6)$ is nonempty only when $N_b < 10$ (else $M_b = N_b$). It is discharged by two finite, reproducible computations.

*(4a)* A direct packed-bitset computation of $L^2(10)$ shows it is gap-free not only on its central block but downward to $L := 740082854 = 0.11439\ldots \cdot P_{10}$ (with $P_{10} = 6469693230$): every integer of $[L, \sigma^2(10)/6]$ lies in $L^2(10)$. Consequently every $a/b$ with $a/b \in [0.1144, B_{10}/6)$ has $aP_{10}/b \in L^2(10)$, hence is representable. As $B_7/6 = 0.12740\ldots \geq 0.1144$, this covers the *entire* sliver of every $b$ with $N_b \geq 7$.

*(4b)* The residual region $a/b \in [B_{N_b}/6, 0.1144)$ forces $N_b \leq 6$, i.e. $b \mid 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$, and is bounded ($a < 0.1144\, b \leq 3434$). An exhaustive search over the $b \mid 30030$ and the finitely many admissible numerators yields exactly 47 coprime pairs $(a, b)$; for each, $aP_N/b \in L^2(N)$ for some $N \in [N_b, 10]$, so each is representable.

Together (4a)+(4b) cover $[B_{N_b}/6, B_{M_b}/6)$, completing the proof for all $a/b \geq B_{N_b}/6$. $\square$

**Remark 1.** For $N_b \geq 10$ the threshold $B_{N_b}/6$ is exactly the onset of the deep regime, so Theorem 2 is sharp: the dichotomy $a/b \geq B_{N_b}/6$ (solved) vs. $a/b < B_{N_b}/6$ (open, §7) has no gap.

**A $b$-uniform threshold.** The threshold $B_{N_b}/6$ grows with the denominator ($\sim (\log\log)^2/12$). We give an *unconditional, $b$-uniform* threshold $\gamma_\infty$ that improves Theorem 2 for all denominators of largest prime factor $\geq 19$. Write $L^1(N) = \{\sum_{i \in T} P_N/p_i : T \subseteq \{1, \ldots, N\}\}$ for the single-prime "ruler". Splitting a subset of $D_2(N)$ according to whether each pair contains $p_N$ gives the exact sumset recurrence

$$L^2(N) = p_N \cdot L^2(N-1) \;+\; L^1(N-1). \tag{3}$$

Let $\gamma_N = \inf\{\gamma : [\gamma P_N,\, \sigma^2(N) - \gamma P_N] \cap \mathbb{Z} \subseteq L^2(N)\}$ be the gap-free floor, and (the ruler residues mod $p_N$ being $P_{N-1} \cdot p_i^{-1}$)

$$w_N = \max_{r \in \mathbb{Z}/p_N} \min\Big\{ \textstyle\sum_{i \in T} \frac{1}{p_i} :\; T \subseteq \{1, \ldots, N-1\},\; \sum_{i \in T} p_i^{-1} \equiv r \pmod{p_N} \Big\}. \tag{4}$$

**Lemma 1** (Floor recurrence)**.** *If the subset sums of $\{p_i^{-1} \bmod p_N : 1 \leq i < N\}$ cover $\mathbb{Z}/p_N$ (so $w_N < \infty$), then $\gamma_N \leq \gamma_{N-1} + w_N/p_N$.*

*Proof.* By the complement symmetry $m \in L^2(N) \Leftrightarrow \sigma^2(N) - m \in L^2(N)$ it suffices to put the lower edge: every integer $m \in [(\gamma_{N-1} + w_N/p_N)P_N,\; \sigma^2(N)/2]$ in $L^2(N)$. By (4) choose $T$ with $\sum_{i \in T} p_i^{-1} \equiv m\, P_{N-1}^{-1}\,(p_N)$ and $\sum_{i \in T} 1/p_i \leq w_N$; then $y := \sum_{i \in T} P_{N-1}/p_i = P_{N-1} \sum_{i \in T} 1/p_i \leq w_N P_{N-1}$ satisfies $y \equiv m\,(p_N)$, so $x := (m - y)/p_N \in \mathbb{Z}$. Now $x \geq (m - w_N P_{N-1})/p_N \geq \gamma_{N-1} P_{N-1}$, and $x \leq m/p_N \leq \sigma^2(N)/(2p_N) = B_N P_{N-1}/2 \leq (B_{N-1} - \gamma_{N-1})P_{N-1} = \sigma^2(N-1) - \gamma_{N-1} P_{N-1}$, the last step from (2) ($B_N \leq 1.06\, B_{N-1}$) and $\gamma_{N-1} \leq 0.14\, B_{N-1}$. Thus $x$ lies in the gap-free range of $L^2(N-1)$, so $x \in L^2(N-1)$, and $m = p_N x + y \in L^2(N)$ by (3). $\square$

**Theorem 3** (Uniform threshold)**.** *Every $a/b$ with $b$ squarefree and $a/b \geq \gamma_\infty$ is $\omega = 2$ representable, where $\gamma_\infty \leq 0.121$ is an explicit absolute constant. Combined with Theorem 2, $a/b$ is representable whenever $a/b \geq \min(B_{N_b}/6,\; \gamma_\infty)$. Unlike the $b$-dependent threshold $B_{N_b}/6 \to \infty$ of Theorem 2, this bound is* uniform in $b$*; it improves Theorem 2 for every $b$ whose largest prime factor is $\geq 17$ (then $B_{N_b}/6 > \gamma_\infty$).*

*Proof.* For each $N$ the subset sums of $\{p_i^{-1} \bmod p_N : i < N\}$ cover $\mathbb{Z}/p_N$: indeed the $\lceil \sqrt{4p_N - 3} \rceil$ largest such inverses are distinct and nonzero, so by Dias da Silva–Hamidoune [7] their subset sums already exhaust $\mathbb{Z}/p_N$ (Lemma 7(iii) gives the same for all $N$); moreover those primes exceed $p_N/2$ once $p_N \geq 16(\log p_N)^2$, whence $w_N \leq \sqrt{4p_N} \cdot \frac{2}{p_N} = 4/\sqrt{p_N}$. Iterating Lemma 1 from the base $\gamma_{12} = 0.08216\ldots$ (an exact packed-bitset computation of the gap-free floor of $L^2(12)$,

$P_{12} = 7420738134810$),

$$\gamma_N \;\le\; \gamma_{12} + \sum_{k=13}^{N} \frac{w_k}{p_k} \;\le\; \underbrace{\gamma_{12}}_{0.0822} + \underbrace{\sum_{k=13}^{2000} \frac{w_k}{p_k}}_{0.0339} + \underbrace{\sum_{k>2000} \frac{4}{p_k^{3/2}}}_{\le 0.005} \;\le\; 0.121.$$

The middle sum is evaluated exactly from (4) by an $O(p_k \cdot k)$ minimum-weight subset (0/1-knapsack) computation over $\mathbb{Z}/p_k$ (the constants $w_k p_k$ stay bounded, decreasing from 9.1 to 2.5); the tail is a convergent prime sum. Hence $\gamma_N \le 0.121$ for all $N \ge 12$. Given $a/b \ge \gamma_\infty$, choose $N \ge N_b$ with $B_N \ge a/b + \gamma_\infty$ (possible since $B_N \to \infty$); then $aP_N/b$ is an integer (Theorem 2, Step 2) lying in $[\gamma_N P_N, \sigma^2(N) - \gamma_N P_N] \subseteq L^2(N)$, so by (1) $a/b$ is representable. Finally $B_{N_b}/6 > 0.121$ exactly when $N_b \ge 7$, i.e. when $b$ has a prime factor $\ge 17$. ☐

## 3. The rational $\omega = 3$ theorem

We now prove the *full* rational $\omega = 3$ statement of Erdős and Graham: every positive rational with squarefree denominator is a finite sum of distinct unit fractions whose denominators are products of three distinct primes (*sphenic*). This is the statement BEG15 attributed to Erdős and Graham but report was “never published”, and about whose truth their three authors record themselves as disagreeing ([2], closing footnote). To our knowledge this is the first complete proof. It uses only the *unconditional* $\omega = 2$ uniform-threshold theorem (Theorem 3 of §2): it never touches the open deep core ($\star$) of §7, so the rational $\omega = 3$ theorem is exactly as secure as Theorem 3.

**Theorem 4** (rational $\omega = 3$)**.** *Every positive rational $a/b$ with $b$ squarefree is a finite sum of distinct unit fractions, each denominator a product of three distinct primes.*

### 3.1. Complementation and the three regimes.

For $k \ge 1$ write $B_k(N) = \sum_{i_1<\cdots<i_k \le N} 1/(p_{i_1} \cdots p_{i_k}) = e_k(1/p_1, \ldots, 1/p_N)$, so $B_2(N) = B_N$. Clearing $1/(p_a p_b p_c)$ by $P_N$ turns it into $P_N/(p_a p_b p_c)$, the product of the *other* $N-3$ primes; hence $a/b$ is $\omega = 3$ representable using primes $\le p_N$ iff $(a/b)P_N$ is a subset sum of $\{P_N/(p_i p_j p_k) : i < j < k \le N\}$, and BEG15's central-block lemma applies.

**Theorem 5** (BEG15, Lemma 1)**.** *For $N \ge 8$, every integer of $[\frac{1}{6}\sigma_3(N), \frac{5}{6}\sigma_3(N)]$ is such a subset sum, where $\sigma_3(N) = B_3(N)P_N$. Equivalently every $a/b$ with $b \mid P_N$ and $a/b \in [B_3(N)/6,\ 5B_3(N)/6]$ is $\omega = 3$ representable.*

This is BEG15's *published* result (their covering uses Olson's theorem [6], not Dias da Silva–Hamidoune); we cite it and do not reprove it. Since $B_3(N+1) = B_3(N) + B_2(N)/p_{N+1} \le 5B_3(N)$ for $N \ge 8$ (the central blocks overlap, exactly as for $\omega = 2$), the union over $N \ge \max(8, N_b)$ gives

$$\text{(5)} \qquad \text{(regime I)} \qquad \big[\, B_3(\max(8, N_b))/6,\ \infty \big) \;\subseteq\; \{\, a/b :\ \omega = 3 \text{ representable} \,\}.$$

In particular every $a/b \ge B_3(8)/6 = 0.04174\ldots$ is covered. What BEG15 left open — “recovering very small rational numbers”, where “the structure of $L_{n+3}(n)$ is not well understood at the ends of the interval” ([2]) — is the *deep* part $a/b < B_3(N_b)/6$. We settle it by a *lift* to the unconditional $\omega = 2$ theorem.

### 3.2. The lift.

**Lemma 2** (lift)**.** *Let $t = a/b > 0$ and let $r$ be a prime with $r \nmid b$. If $V := t\,r$ is $\omega = 2$ representable using a prime set $\Pi$ with $r \notin \Pi$, say $V = \sum_{(i,j)} 1/(q_i q_j)$ with $q_i, q_j \in \Pi$, then $t = \sum_{(i,j)} 1/(q_i q_j r)$ is a sum of distinct sphenic unit fractions.*

*Proof.* $t = V/r = \sum 1/(q_i q_j r)$. The three primes $q_i, q_j, r$ are distinct because $r \notin \Pi$, and distinct pairs give distinct denominators. Choosing $\Pi \cup \{r\} \subseteq \{p_1, \ldots, p_M\}$ with $b \mid P_M$ makes every term integral over $b$ (then $V \cdot \prod \Pi = aP_M/b \in \mathbb{Z}$ as $\gcd(a, b) = 1$). ☐

Thus $t$ is covered once $V = tr$ lies in the gap-free range of an $\omega = 2$ cover that omits $r$. Write $\Pi(M,r) = \{p_1,\dots,p_M\} \setminus \{r\}$, $B_\Pi(M,r) = \sum_{i<j:\, p_i,p_j\in\Pi} 1/(p_ip_j)$, and let $\gamma_{\rm ex}(M,r)$ be the $\omega = 2$ gap-free floor of the cover over $\Pi(M,r)$ — the analogue of $\gamma_N$ (§2) with one ruler prime deleted. By the mechanism of Theorem 3 the cover is gap-free on $[\gamma_{\rm ex}(M,r)\,P_\Pi,\ \sigma - \gamma_{\rm ex}(M,r)\,P_\Pi]$, so the lift covers

$$t \in \Big[\ \tfrac{\gamma_{\rm ex}(M,r)}{r},\ \tfrac{B_\Pi(M,r)-\gamma_{\rm ex}(M,r)}{r}\ \Big] =: \big[\,\ell(M,r),\ R(M,r)\,\big].$$

Two uses. **Regime III** (no exclusion): $\Pi = \{p_1,\dots,p_M\}$, $r = p_{M+1}$, floor $\gamma_M \le 0.121$ (Theorem 3), so $\ell = \gamma_M/p_{M+1} \to 0$; at $M = 10$ the interval is $[\gamma_{10}/p_{11}, (B_{10}-\gamma_{10})/p_{11}] = [0.0037, 0.0271]$ and the union over $M \ge 10$ is $(0, 0.0271]$. **Regime II** (exclude one prime $r$): the bridge, controlled by $\gamma_{\rm ex}(M,r)$ via Lemma 3.

3.3. **Floor-robustness: deleting one ruler prime.** The one genuinely new estimate is that deleting a prime from the $\omega = 2$ ruler raises the floor by a bounded amount, uniformly in the level $M$ and in which prime is deleted. Recall the covering cost (4) of §2 driving $\gamma_k \le \gamma_{k-1} + w_k/p_k$. Deleting ruler prime $p_s$ ($s < k$) gives the *excluded cost*

$$w_k^{\rm ex}(s) = \max_{t\in\mathbb{Z}_{p_k}}\ \min\Big\{\sum_{i\in T}\tfrac{1}{p_i} : T \subseteq \{1,\dots,k-1\}\setminus\{s\},\ \sum_{i\in T} p_i^{-1} \equiv t \pmod{p_k}\Big\}, \tag{6}$$

and the recurrence $\gamma_{\rm ex}(k,r) \le \gamma_{\rm ex}(k-1,r) + w_k^{\rm ex}(s)/p_k$, valid whenever the punctured ruler still covers $\mathbb{Z}_{p_k}$ (so $w_k^{\rm ex} < \infty$).

**Lemma 3** (floor-robustness, uniform in $r$ and $M$)**.** *For every prime $r$ and every $M \ge 11$, $\gamma_{\rm ex}(M,r) \le 0.190$.*

*Proof.* Work at level $M \ge 11$ (free, since $b \mid P_{10} \Rightarrow b \mid P_{11}$); this matters because the $M = 10$ small-prime floors are large ($\gamma_{\rm ex}(10,5) = 0.165$) yet all drop by $M = 11$. Bound $\gamma_{\rm ex}(M,r) \le \text{base} + \sum_{k=12}^{M} w_k^{\rm ex}/p_k$ piecewise.

*Base.* Exactly (packed bitset; Table 1), $\max_{r\le p_{11}} \gamma_{\rm ex}(11,r) = 0.116$, attained at $r = 19$; for $r \ge p_{12}$ the base is the full-ruler $\gamma_{11} = 0.0957$. So base $\le 0.116$. (The floor is *non-monotone* in $r$, so the small primes $r = 2,3,5,7$ must be — and are — included in this maximum.)

*Exact cost,* $12 \le k \le 220$. With $w_k^{\max} := \max_{1\le s<k} w_k^{\rm ex}(s)$ the worst single-prime deletion, $\sum_{k=12}^{220} w_k^{\max}/p_k = 0.0498$ (worst single level $w_kp_k = 14.32$ at $k = 12$), and the punctured ruler was verified to cover $\mathbb{Z}_{p_k}$ for *every* deletion $s < k$ at every such $k$ (zero failures).

*Tail,* $k > 220$. The deletion crossover is $k_0 = 120$: for $k \ge 120$ the $\lceil\sqrt{4p_k-3}\rceil + 1$ largest ruler primes all exceed $p_k/2$, so after one deletion at least $\sqrt{4p_k-3}$ of them remain $> p_k/2$, each of inverse-weight $< 2/p_k$; by Dias da Silva–Hamidoune [7] their subset sums already exhaust $\mathbb{Z}_{p_k}$, whence $w_k^{\rm ex} \le \sqrt{4p_k-3}\cdot 2/p_k < 4/\sqrt{p_k}$. As $220 > k_0$,

$$\sum_{k>220} w_k^{\rm ex}/p_k \ \le\ \sum_{k>220}\frac{4}{p_k^{3/2}}\ =\ 0.024$$

(a convergent prime sum, evaluated to convergence). Summing, $\gamma_{\rm ex}(M,r) \le 0.116+0.0498+0.024 = 0.190$ for every prime $r$ and every $M \ge 11$. □

**Remark 2.** The bound is uniform over *which* prime is deleted because it uses the worst-case cost $w_k^{\max}$, which dominates every single-$r$ value; the actual per-$r$ floors are smaller ($\gamma_{\rm ex}(M,r) \le 0.160$ for all $r$, worst at $r = 19$; Table 1). Only the uniform 0.190 is load-bearing in the seam.

3.4. **The seam.**

*Proof of Theorem 4.* Fix squarefree $b$. By (5) every $a/b \ge B_3(\max(8,N_b))/6$ is $\omega = 3$ representable (regime I). It remains to cover the deep part $0 < a/b < B_3(\max(8,N_b))/6$.

Choose a prime $r$ with $11 \le r$ and $r \nmid b$ (only finitely many primes divide $b$, so such $r$ exists). By Lemma 3, $\gamma_{\rm ex}(M,r) \le 0.190$ for all $M \ge 11$, so the regime-II interval $[\ell(M,r), R(M,r)]$ satisfies

$$\ell(M,r) = \frac{\gamma_{\rm ex}(M,r)}{r} \le \frac{0.190}{11} < 0.0173, \qquad R(M,r) = \frac{B_\Pi(M,r) - \gamma_{\rm ex}(M,r)}{r} \xrightarrow[M\to\infty]{} \infty$$

(as $B_\Pi(M,r) \to \infty$). Consecutive levels overlap, since $B_\Pi(M,r) - \gamma_{\rm ex}(M,r) \ge 0.6 \ge \gamma_{\rm ex}(M{+}1,r)$; hence the union over $M \ge \max(N_b, \pi(r))$ of regime-II intervals is $[\ell_*, \infty)$ with $\ell_* \le 0.0173$. Combined with regime III's $(0, 0.0271]$ and $0.0173 < 0.0271$, the deep part is covered: $(0, 0.0271] \cup [0.0173, \infty) = (0, \infty)$. Therefore every $a/b > 0$ is $\omega = 3$ representable. $\square$

**Remark 3** (explicit bridge)**.** For $b$ coprime to 11 one may bridge $[0.0271, 0.0417]$ with $r = 11$ at $M = 10$ directly: $\gamma_{\rm ex}(10, 11) = 0.14605$ and $B_\Pi(10, 11) = 0.822$ give $[\ell, R] = [0.0133,\ 0.0615]$, overlapping both regime III $(0.0133 \le 0.0271)$ and regime I $(0.0417 \le 0.0615)$. For $b$ divisible by 11 the smallest admissible $r$ is larger and Lemma 3 applies as above; the floor is non-monotone in $r$ (worst at $r = 19$), which is why a uniform bound, not a "large $r$ is easier" heuristic, is required.

3.5. **Numerical certificate.** The load-bearing constants of Lemma 3 are exact finite computations: packed-bitset gap-free floors $\gamma_{\rm ex}(11, r)$, and an $O(p_k k)$ minimum-weight subset (0/1-knapsack) computation over $\mathbb{Z}_{p_k}$ for the cost (6). They are calibrated against Theorem 3: the full-ruler run reproduces $\sum_{13}^{2000} w_k/p_k = 0.03388$ and $w_{13}p_{13} = 9.14$.

| $r$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma_{\rm ex}(11, r)$ | .0866 | .0938 | .1044 | .1086 | .1047 | .1063 | .1111 | **.1160** | .1088 | .1081 | .1144 |

TABLE 1. Exact base floors at $M = 11$ (packed bitset). Maximum 0.116 at $r = 19$. Assembled uniform bound: $\gamma_{\rm ex}(M,r) \le \underbrace{0.116}_{\text{base}} + \underbrace{0.0498}_{\sum_{12}^{220} w_k^{\max}/p_k} + \underbrace{0.024}_{\sum_{k>220} 4p_k^{-3/2}} = 0.190$ for all primes $r$, all $M \ge 11$ (worst single level $w_k p_k = 14.32$ at $k = 12$; DSH crossover $k_0 = 120$). Regime-II reach, e.g. $R(10, 11) = (0.822 - 0.14605)/11 = 0.0615$, $R(10, 13) = (0.841 - 0.15042)/13 = 0.0531$.

## 4. Density one

**Theorem 6** (Density one)**.** *For every squarefree $b$, the set of $\omega = 2$ representable $a/b$ has natural density* 1 *in* $\{a/b : a \in \mathbb{N},\ \gcd(a,b) = 1\}$.

*Proof.* For fixed $b$ the threshold $B_{N_b}/6$ is a fixed number; by Theorem 2 every $a/b \ge B_{N_b}/6$ is representable, so at most the finitely many $a < b\, B_{N_b}/6$ can fail. Hence density 1. $\square$

## 5. Deep almost-all, conditional on a two-point estimate

Write $D_1(N) = \{P_N/p_i : i \le N\}$ for the "ruler" and $L^1(N)$ for its subset sums; $|L^1(N)| = 2^N$ and these sums are distinct (geometrically separated). Deleting the largest prime gives the exact recurrence

$$L^2(N) = p_N \cdot L^2(N-1) + L^1(N-1), \tag{7}$$

since the atoms of $D_2(N)$ split as $\{p_N \cdot P'_N/(p_i p_j) : i < j \le N-1\}$ (scaled $D_2(N-1)$) together with $\{P_N/(p_i p_N) = P'_N/p_i : i \le N-1\} = D_1(N-1)$, where $P'_N = P_{N-1}$. For a target $m$ put

$$R_N(m) = \#\{s \in L^1(N-1) : s \equiv m \pmod{p_N},\ (m-s)/p_N \in L^2(N-1)\},$$

so by (7), $m \in L^2(N) \iff R_N(m) \geq 1$. Let $\rho_N(\theta)$ be the density of $L^2(N)$ in a window at scale $\theta P_N$, write $E_N = \mathbb{E}_m R_N$ (mean over the window), and set

$$\nu(d) = \#\{(s,s') \in L^1(N-1)^2 : s'-s = d\,p_N\}, \quad g_{N-1}(d) = \frac{C_{N-1}(d)}{\rho_{N-1}^2}, \quad C_{N-1}(d) = \Pr_x[x, x-d \in L^2(N-1)].$$

**Conjecture 1** (($\star_2$): an explicit, Hardy–Littlewood–free two-point bound)**.** *For each fixed deep scale,* $\mathbb{E}_{d\sim\nu}\big[g_{N-1}(d)\big] = \dfrac{\sum_{d\neq 0} \nu(d)\, g_{N-1}(d)}{\sum_{d\neq 0}\nu(d)} \leq 1 + o(1)$ *as* $N \to \infty$.

**Theorem 7** (Deep almost-all, conditional on ($\star_2$))**.** *Assume* ($\star_2$)*. Then for every fixed deep scale* $\theta > 0$, $\rho_N(\theta) \to 1$*: almost every deep rational* $a/b$ *(with* $b \mid P_N$ *and* $a/b \approx \theta$*) is* $\omega = 2$ *representable.*

*Proof.* The following are elementary counting identities (each exact; verified for $N \leq 11$).

*(I1) First moment.* Summing $R_N(m)$ over a scale-$\theta$ window, $\sum_m R_N(m) = \sum_{s\in L^1(N-1)} \#\{x \in L^2(N-1) : p_N x + s \text{ in window}\} = 2^{N-1}\rho_{N-1}(\theta)\cdot(\text{window}/p_N)$, since each of the $|L^1(N-1)| = 2^{N-1}$ shifts $s$ contributes the $L^2(N-1)$-points in the rescaled window. Dividing by the window length,

$$E_N = T_N\, \rho_{N-1}(\theta), \qquad T_N = \frac{2^{N-1}}{p_N}.$$

*(I2)–(I3) Second moment.* The factorial second moment is $\Delta_N := \mathbb{E}_m[R_N(R_N-1)] = p_N^{-1}\sum_{d\neq 0}\nu(d)\, C_{N-1}(d)$ (pairs of distinct ruler hits $s, s'$ with $s' - s = d\,p_N$, weighted by the chance both $(m-s)/p_N, (m-s')/p_N \in L^2(N-1)$). Hence $\mathrm{Var}_m(R_N)/E_N^2 = 1/E_N + \delta_N$ with $\delta_N := \Delta_N/E_N^2 - 1$, and a direct computation (using $\sum_{d\neq 0}\nu(d) = 2^{2(N-1)}/p_N\,(1+o(1))$ from the equidistribution of the ruler mod $p_N$) gives

$$\delta_N = \mathbb{E}_{d\sim\nu}\big[g_{N-1}\big] - 1. \tag{8}$$

*Conclusion.* By Chebyshev, $1-\rho_N(\theta) = \Pr_m[R_N(m) = 0] \leq \mathrm{Var}_m(R_N)/E_N^2 = 1/E_N + \delta_N$. Above the (descending) floor $\rho_{N-1}$ is bounded below, so $E_N = T_N\rho_{N-1} \to \infty$ (as $T_N = 2^{N-1}/p_N \to \infty$). Hence *if* $\delta_N \to 0$ — which by (8) is exactly ($\star_2$) — then $1-\rho_N(\theta) \leq 1/E_N + \delta_N \to 0$, i.e. $\rho_N(\theta) \to 1$. (Only this one direction is asserted: the upper Chebyshev bound gives ($\star_2$) $\Rightarrow \rho_N \to 1$, which is what deep-almost-all requires; we do not claim the converse.) ☐

**Remark 4** (($\star_2$) is a clean Fourier-energy statement)**.** With $u = \mathbf{1}_{L^2(N-1)} - \rho_{N-1}$ and $\tilde\nu(\xi) \geq 0$ the ruler autocorrelation transform, one has the exact identity $\mathbb{E}_{d\sim\nu}[g_{N-1}] - 1 = \rho_{N-1}^{-2}\sum_{\xi\neq 0}|\hat u(\xi)|^2\,\tilde\nu(\xi)$, both factors $\geq 0$. Thus ($\star_2$) says precisely that the $\nu$-weighted Fourier energy of the $L^2$-indicator is $o(\rho_{N-1}^2)$ — no prime pairs, Bateman–Horn, or Kloosterman sums enter.

## 6. Supporting lemmas (all unconditional)

These lemmas are not used in Theorems 2–7; they describe the circle-method infrastructure of the open core (§7) and show in particular that the cancellation (minor-arc) side is elementary and Hardy–Littlewood–free. Throughout $Q = P_N$, $a_{ij} = P_N/(p_i p_j)$, $\|x\| = \mathrm{dist}(x, \mathbb{Z})$, and $G(\xi) = \prod_{i<j}|\cos(\pi\xi a_{ij}/Q)|$ is the modulus of the (normalized) subset-sum exponential sum.

**Minor-arc bound.**

**Lemma 4** (Cosine convexity)**.** *For all* $\theta \in \mathbb{R}$, $\log|\cos\pi\theta| \leq -2\|\theta\|^2$.

*Proof.* Both sides are 1-periodic and even, so we may take $t := \|\theta\| \in [0, \frac{1}{2}]$. Since $|\cos\pi\theta|^2 = \frac{1}{2}(1+\cos 2\pi t) = 1 - \sin^2(\pi t)$, it suffices to show $\sin^2(\pi t) \geq 4t^2$, i.e. $\sin\pi t \geq 2t$ on $[0, \frac{1}{2}]$. The two sides agree at $t = 0$ and $t = \frac{1}{2}$, and $\sin\pi t$ is concave on $[0, \frac{1}{2}]$ while $2t$ is linear; hence $\sin\pi t \geq 2t$ there. Thus $|\cos\pi\theta|^2 \leq 1 - 4t^2 \leq e^{-4t^2}$, and taking logarithms gives the claim. ☐

**Lemma 5** (Structural identity and energy bound)**.** *$\|\xi a_{ij}/Q\| = \|\xi/(p_i p_j)\|$, so $G(\xi) = \prod_{i<j} |\cos(\pi\xi/(p_i p_j))|$ and, by Lemma 4,*

$$\log G(\xi) \;\le\; -2\,E(\xi), \qquad E(\xi) := \sum_{1\le i<j\le N} \big\|\xi/(p_i p_j)\big\|^2.$$

*Proof.* $a_{ij}/Q = 1/(p_i p_j)$, so $\xi a_{ij}/Q \equiv \xi/(p_i p_j) \pmod 1$ and the moduli of the fractional parts coincide; the rest is Lemma 4 summed over $i<j$. ☐

Define the major arcs $\mathfrak{M} = \bigcup_{q'\le p_N} \bigcup_{(a,q')=1} \{\xi : \|\xi/Q - a/q'\| \le \eta/q'\}$ ($\eta$ a small constant) and the minor arcs $\mathfrak{m} = \mathbb{Z}/Q \setminus \mathfrak{M}$. The vanishing of $\|\xi/(p_i p_j)\|$ is purely combinatorial:

**Lemma 6** (Exact divisibility)**.** *$\|\xi/(p_i p_j)\| = 0 \iff p_i p_j \mid \xi \iff p_i, p_j \in S(\xi) := \{p \le p_N : p \mid \xi\}$. Hence the number of vanishing atoms is $\binom{|S(\xi)|}{2}$; if $P_N \nmid \xi$ then $|S(\xi)| \le N-1$, so at least $\binom{N}{2} - \binom{N-1}{2} = N-1$ atoms have $\|\xi/(p_i p_j)\| > 0$.*

Write $\#\mathrm{far}_\delta(\xi) = \#\{(i,j) : \|\xi/(p_i p_j)\| \ge \delta\}$. The energy bound expresses $G$ through the far-count with a fully rigorous, elementary step:

**Corollary 1.** *For every $\delta \in (0, \frac{1}{2}]$ and every $\xi$, $G(\xi) \le \exp\!\big(-2\delta^2\,\#\mathrm{far}_\delta(\xi)\big)$.*

*Proof.* $E(\xi) \ge \delta^2\,\#\mathrm{far}_\delta(\xi)$ by definition; apply Lemma 5. ☐

Thus the minor-arc estimate reduces to a far-count lower bound on $\mathfrak{m}$. We record this as a proposition rather than a theorem: the qualitative mechanism is elementary and Hardy–Littlewood–free, but the quantitative *positive-proportion* far-count rests on a simultaneous-Diophantine (large-sieve type) estimate that we do not carry out here; its explicit constants are certified numerically in the companion note [11]. *This proposition is not used in Theorems 2–7.*

**Proposition 1** (Positive-proportion far-count; large-sieve estimate)**.** *There exist $\delta, \varepsilon, c > 0$ (depending only on $\eta$) with the following property. For all $\xi \in \mathfrak{m}$, $\#\mathrm{far}_\delta(\xi) \ge \varepsilon\binom{N}{2}$, and hence by Corollary 1, $G(\xi) \le \exp\!\big(-c\binom{N}{2}/\log p_N\big)$ — superexponential in $N$, with no prime-pair, Bateman–Horn, or Kloosterman input.*

**Remark 5** (Status and the elementary partial result)**.** The exact part of the far-count is rigorous (Lemma 6: off $P_N \mid \xi$, at least $N-1$ atoms are nonzero). The proposition's *positive-proportion* strengthening is the contrapositive of "if $\ge \varepsilon\binom{N}{2}$ of the distinct semiprimes $p_i p_j$ satisfy $\|\xi/(p_i p_j)\| < \delta$, then the simultaneous near-congruences force $\xi/Q$ into an $O(1/q')$-window of some $a/q'$, $q' \le p_N$, i.e. $\xi \in \mathfrak{M}$". This is a standard-shaped but nontrivial large-sieve estimate (the two-term Bezóut reduction is insufficient because the atoms have size $\sim P_N$); it is certified for $\delta = 0.1$, $\varepsilon \approx 0.3$ and the sharp $1/\log p_N$ exponent in [11], and is stated here as a proposition pending a written proof. The point relevant to this paper is qualitative and is *unconditional*: the cancellation side of the circle method is governed by the elementary, HL-free energy $E(\xi) = \sum \|\xi/(p_i p_j)\|^2$, with no Kloosterman sums anywhere.

**Residue covering.**

**Lemma 7** (Residue covering is Kloosterman-free)**.** *For every $q \le p_N$ dividing $P_N$, $L^2(N) \bmod q$ is the set of subset sums of the scaled prime-inverses $\{A_q\, p_j^{-1} \bmod q : p_j \nmid q\}$ ($A_q = (P_N/q) \bmod q$); it equals all of $\mathbb{Z}/q$. The count bias $\mathrm{rb}(a,q) = \prod_{(i,j)\text{ active}} |\cos(\pi a\, a_{ij}/q)|$ is:* (i) *0 for every even $q$ and primitive $a$;* (ii) *geometric for fixed odd $q$;* (iii) *$\to 0$ for large prime $q$.*

*Proof.* (i) Write $q = 2q'$, $q'$ odd. The atom $a_{1j} = P_N/(2p_j)$ (with $p_j \nmid q$) is odd and divisible by $q'$, so $a_{1j} \equiv q' = q/2 \pmod q$. For primitive $a$ (hence odd, as $2 \mid q$), the corresponding factor $1 + e(a\,a_{1j}/q) = 1 + e(a/2) = 1 + (-1)^a = 0$. One such atom annihilates the product, so $\mathrm{rb}(a,q) = 0$. *(ii)* For fixed odd $q$ the per-factor magnitude is $\le 2\cos(\pi/q) < 2$, giving geometric decay (e.g. $q = 3$:

$|\Phi| \equiv 1$, bias $2^{-(N-2)}$; $q = 5$: bias $\varphi^{n_+ - n_-}/2^{N-1}$, $\varphi$ the golden ratio); the elementary computation is in [12]. *(iii)* For $q = p_t$ a large prime, $L^2(N) \bmod q$ is the set of subset sums of the family $S = \{A_q p_j^{-1} \bmod q : p_j \nmid q\}$. The number of representations of $m$ as a subset sum is

$$R(m) = \frac{1}{q} \sum_{t \in \mathbb{Z}/q} \prod_{s \in S} \big(1 + e(ts/q)\big)\, e(-tm/q),$$

so $|R(m) - 2^{|S|}/q| \le q^{-1} \sum_{t \neq 0} \prod_{s \in S} |1 + e(ts/q)| = 2^{|S|} q^{-1} \sum_{t \neq 0} \prod_{s \in S} |\cos(\pi ts/q)|$. Hence *if*

$$\sum_{t \neq 0} \prod_{s \in S} \big|\cos(\pi ts/q)\big| \;<\; 1, \tag{9}$$

then $R(m) > 0$ for every $m$, i.e. $S$ covers $\mathbb{Z}/q$. To verify (9), bound the product through the energy: by Lemma 4,

$$\prod_{s \in S} |\cos(\pi ts/q)| \;\le\; \exp\Big(-2 \sum_{s \in S} \|ts/q\|^2\Big).$$

Fix $t \neq 0$. A power-saving bound for the prime-inverse sums $\sum_{p \le q} e(u\,\overline{p}/q)$ (Fouvry–Shparlinski, Baker, Irving, Korolev–Changa; [10]) — a *long*-sum input in the range $[q^{1/2+\varepsilon}, q^{3/2}]$, *not* the beyond-GRH short Kloosterman regime — says the inverses $\{\,\overline{p_j} \bmod q\,\}$ are equidistributed; in particular at most $o(|S|)$ of them fall in any fixed-length window, so for some absolute $\delta_0 > 0$ at most $\frac{1}{2}|S|$ of the $ts/q = tA_q\overline{p_j}/q$ have $\|ts/q\| < \delta_0$. Hence $\sum_{s \in S} \|ts/q\|^2 \ge \frac{1}{2}\delta_0^2 |S| = cN + O(1)$, giving $\prod_{s \in S} |\cos(\pi ts/q)| \le e^{-2cN}$. Since $q \le p_N$ and $N \gg \log p_N$, $\sum_{t \neq 0} \prod_s |\cos(\pi ts/q)| \le q\, e^{-2cN} < 1$ for $N$ large, so (9) holds and $S$ covers $\mathbb{Z}/q$. The bias estimates (i),(ii) refine this for small $q$. ☐

**Coarse-lattice covering.**

**Lemma 8** (Coarse-lattice covering)**.** *Let $F_0 = \{p \le y\}$ and $D = \prod_{p \in F_0} p$, with $y = o(p_N)$. For all sufficiently large $N$, every residue $r \bmod D$ is the value of a subset of $\{\, P_N/(p\,p_j) : p \in F_0,\ p_j \nmid D \,\}$ of total value at most $C_F$, with $C_F/P_N \to 0$.*

*Proof.* By CRT $\mathbb{Z}/D \cong \prod_{p \in F_0} \mathbb{Z}/p$. An atom $P_N/(p\,p_j)$ $(p \in F_0,\ p_j \nmid D)$ is $\equiv 0 \pmod{p'}$ for every $p' \in F_0 \setminus \{p\}$ (its product $\prod_{l \neq p,j} p_l$ contains $p'$) and $\not\equiv 0 \pmod p$; hence it moves only the $p$-coordinate, and the coordinates may be covered independently. Fix $p \in F_0$. The map $p_j \mapsto A_p\, p_j^{-1} \bmod p$ is injective on residues mod $p$ ($A_p$ is a unit and inversion is a bijection of $(\mathbb{Z}/p)^\times$), so primes $p_j$ in distinct classes mod $p$ give distinct nonzero residues. For $N$ large the non-free primes $p_j \le p_N$ occupy all $p - 1$ nonzero classes mod $p$ (prime equidistribution, since $p \le y = o(p_N)$), so among them are at least $\sqrt{4p-3}$ atoms with *distinct* nonzero residues mod $p$; by the Dias da Silva–Hamidoune theorem [7] (a set of at least $\sqrt{4p-3}$ distinct nonzero elements of $\mathbb{Z}/p$ has every element of $\mathbb{Z}/p$ as a subset sum) their subset sums fill $\mathbb{Z}/p$. Choosing such a covering set among the atoms with the largest admissible $p_j$ (smallest value, each $\le (1 + o(1))P_N/(p\,p_N)$ when the number used is $o(N)$), coordinate $p$ is covered at value $\le C_p$ with $\sum_{p \in F_0} C_p/P_N \to 0$ by Mertens. We do not optimize the constant; this lemma is not used in Theorems 2–7. ☐

## 7. The open core, and why it is anti-concentration not Hardy–Littlewood

Write $\gamma_N = \inf\{\gamma : [\gamma P_N,\, \sigma^2(N) - \gamma P_N] \cap \mathbb{Z} \subseteq L^2(N)\}$ for the (true) gap-free floor.

**Conjecture 2** (($\star$), the full close)**.** *$\gamma_N \to 0$ as $N \to \infty$. Equivalently $f_N(m) \ge 1$ for every bulk $m$, where $f_N(m) = \#\{S \subseteq D_2(N) : \sum S = m\}$; equivalently the sub-Poisson tail bound $\Pr_m[R_N(m) = 0] \le e^{-c_N E_N}$ holds with $c_N \gg N^2/2^N$.*

**Proposition 2** (($\star$) $\Rightarrow$ full close)**.** *If ($\star$) holds then every positive rational $a/b$ (squarefree $b$) is $\omega = 2$ representable: by Remark 6 it suffices that $a/b \ge \gamma_N$ for some $N \ge N_b$, which $\gamma_N \to 0$ provides.*

**Remark 6** (Rate $\perp$ existence)**.** For fixed $a/b$, $N$ is unbounded (as $b \mid P_N$ for all $N \geq N_b$); hence $\gamma_N \to 0$ at *any* rate yields representability. The doubly-exponential size of certificates reflects the rate, not an obstruction.

**Remark 7** (HL-dissolution)**.** The naive "bottom edge $\equiv$ Hardy–Littlewood prime pairs" reading is a method artifact of gadget/grid constructions, which parametrize a target into a coupled prime pair. The direct route (1) hits $a/b$ as a single subset sum and forms no prime pair; correspondingly (Lemma 7) no $q \sim \sqrt{Q}$ minor arc arises. The genuine obstruction is the *anti-concentration* of the explicit subset-sum set $L^2(N)$, i.e. ($\star$).

**Remark 8** (The additive-energy reduction; open)**.** The two-point hypothesis ($\star_2$) reduces, on the residue side, exactly to the randomness of the 4-fold additive energy of the prime inverses $\{p^{-1} \bmod q : p < q \text{ prime}\}$ (a two-point correlation is a fourth moment). This is already *open*: Bourgain–Garaev [9] give only the upper bound $E^{\mathrm{inv}}(I) \ll |I|^{8/3} + |I|^4/p$, which carries a genuine $|I|^{2/3}$ excess over the random value; the prime-inverse sum bound $|S(t)| < q^{15/16}$ (Fouvry–Shparlinski / Korolev / Irving) is, through the fourth-moment identity, too weak by $q^{7/8}$. A power-saving fourth moment on primes (a triple-Kloosterman cancellation) would suffice, but is not in the literature. The *full* close ($\star$) is strictly stronger: beyond this fourth-moment (two-point) input it requires the *all-order* value-side anti-concentration of $L^2(N)$ — the control of $f_N(m)$ for *every* $m$, not merely on average — which is the genuine remaining content and is not implied by any single moment.

**Proposition 3** (No structural witness; a finite counterexample-witness is again ($\star_2$))**.** *Write $f_N(m) = \#\{S \subseteq D_2(N) : \sum S = m\}$ (Conjecture 2), $|D_2(N)| = \binom{N}{2}$, and $\mu_r = \sum_{m\equiv r\,(q)} f_N(m)$ for the weighted mass of the residue class $r \bmod q$.* (a) Unconditional. *For every fixed $q \geq 2$ and all $N$ with every prime factor of $q$ at most $p_N$,*

$$\max_{r \bmod q}\Big|\mu_r - \frac{2^{\binom{N}{2}}}{q}\Big| \;\leq\; \frac{2^{\binom{N}{2}}}{q}\sum_{k=1}^{q-1}\prod_{1\leq i<j\leq N}\big|\cos(\pi k\, a_{ij}/q)\big| \;\leq\; \frac{2^{\binom{N}{2}}}{q}\,(q-1)\cos(\pi/q)^{\,N-1} \;\xrightarrow[N\to\infty]{}\; 0,$$

*with exactly $N-1$ nontrivial factors when $q = p_l$ is prime (the atoms $a_{lj}$, $j \neq l$; cf. Lemma 7). Hence every residue class mod every fixed $q$ carries mass $(1+o(1))2^{\binom{N}{2}}/q > 0$:* no fixed congruence $m \equiv r\,(q)$, and by CRT no finite covering system, consists entirely of non-representable rationals. *The structural form of a finite witness is excluded unconditionally.* (b) The deep tail is ($\star_2$). *A witness need only be hole-valued on a deep window $W = [\theta\sigma^2(N),\,]$; by Cauchy–Schwarz,*

$$\#\{m \equiv r,\ m \in W :\ f_N(m) \geq 1\} \;\geq\; \frac{\big(\sum_{m\equiv r,\,W} f_N(m)\big)^2}{\sum_{m\equiv r,\,W} f_N(m)^2},$$

*so no class is entirely hole on $W$ once the windowed per-class second moment satisfies $\Delta_N \leq E_N^2(1+o(1))$ — which is exactly ($\star_2$). Thus seeking a finite witness for a counterexample reduces to the same two-point estimate; there is no shortcut.*

*Proof.* (a) For a uniform random $S \subseteq D_2(N)$, $2^{-\binom{N}{2}}\sum_m f_N(m)e(km/q) = \mathbb{E}\,e(k\sum S/q) = \prod_{i<j}\frac{1}{2}(1+e(ka_{ij}/q))$, of modulus $\prod_{i<j}|\cos(\pi k a_{ij}/q)|$; Parseval over $r$ gives the first inequality. As in Lemma 7, $a_{ij} = \prod_{m\neq i,j} p_m$ is coprime to a prime $p_l$ iff $l \in \{i,j\}$, so for $q = p_l$ only the $N-1$ atoms $a_{lj}$ contribute a factor $\neq 1$, each $\leq \cos(\pi/p_l)$; for general fixed $q$, reducing $k/q$ to lowest terms and using the $N-1$ atoms attached to the smallest prime factor of the reduced denominator gives $\prod \leq \cos(\pi/q)^{N-1}$, whence the stated bound and the corollary on class masses. (b) Immediate from Cauchy–Schwarz; the numerator is the squared windowed mass (positive, first-moment asymptotics $f_N \sim E_N$ on $W$) and the denominator is the windowed second moment, i.e. $\Delta_N$. *(Numerics: the exact class masses, $N \leq 12$, confirm $\max_r|\mu_r - 2^{\binom{N}{2}}/q|/(2^{\binom{N}{2}}/q) \leq \sum_{k\neq 0}\prod|\cos|$ with geometric decay; $q = 2, 4$ are exactly equidistributed.)* ∎

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